Republique Algérienne Democratique et Populaire
Ministere de l'Enseignement Superieur
et de la Recherche Scientifique

Université Echahid
Hama Lakhdar El-Oued

Faculté: Sciences Exactes

Départment: Mathématiques

Option: Mathématiques Appliqués

Ordre des numéros:

Thèse de doctorat en sciences

Option: Mathématiques Appliqués

Thème:

Analyse variationnelle et numérique de quelques problèmes aux limites de contact entre des corps

Présentée par:

Mohammed Said Ferhat

Soutenue Publiclement le: 07/01/2023

Membres du Jury:

M. Beggas Mohamed	MCA	Univ. d'El Oued	Président
M. Aissaoui Adel	Professeur	Univ. d'El Oued	Rapporteur
M. Hariz Bekkar Lourabi	MCA	Univ. d'El Oued	Examinateur
M. Menacer Tidjani	Professeur	Univ. de Biskra	Examinateur
M. Djenaihi Youcef	MCA	Univ. de Setif 1	Examinateur
M. Dalah Mohamed	Professeur	Univ. de Constantine 1	Examinateur

Annee Universitaire : 2022/2023

People's Democratic Republic of Algeria
Ministry of Higher Education
and Scientific Research

University Echahid
Hama Lakhdar El Oued

Faculty: Exact Sciences

Department: Mathematics

Option: Applied Mathematics

Number Order:

Thesis for the degree of doctor in sciences

Option: Applied Mathematics

Theme:

Variational and numerical analysis of some problems at the contact limits between bodies

Presented by:

Mohammed Said Ferhat

Publicly presented on: 07/01/2023

Committee Members:

M. Beggas Mohamed	Lecturer "A"	Univ. of El Oued	Chairman
M. Aissaoui Adel	Professor	Univ. of El Oued	Supervisor
M. Hariz Bekkar Lourabi	Lecturer "A"	Univ. of El Oued	Examiner
M. Menacer Tidjani	Professor	Univ. of Biskra	Examiner
M. Djenaihi Youcef	Lecturer "A"	Univ. of Setif 1	Examiner
M. Dalah Mohamed	Professor	Univ. of Constantine 1	Examiner

University Year: 2022/2023

Dedicate

For the sake of Allah, my Creator and my Master, this thesis is dedicated. To our teacher and messenger Muhammad (May Allah bless and grant him) who showed us the purpose of life.

I dedicate this thesis to all those who studied me, to all those I studied and to all our dear colleagues in study and teaching.

To my fantastic mother, who never stops giving in so many ways, and to my dear wife, who supported me in pursuing my dreams and completing my dissertation.

To all of my family, relatives, who represent love and giving, to my cherished brothers and sisters. I dedicate this study to everyone in my life who has touched my heart.

To my dearly loved kids: Adem, Aridj, Imen, Allaa Arrahman, Ahmed Azeddine, Mohammed Lakhdar and Milad, my love for them is growing day by day. To My friends who motivate and assist me.

I dedicate this thesis to a special group of people I love to my heart, who still mean a lot to me. They are no longer a part of this planet, yet their memories still guide my life.

First and foremost, to my father Lakhdar, whose love for me knows no bounds and who taught me the importance of working hard.

I would also like to remember Khalifa Souid, Belkacem Belaid, Djamel Merigua, Rachid Berrouba, Mahmoud Bey and Mahmoud Garbi, from whom I learned a lot.

Mohammed Said Ferhat

Acknowledgements

First and foremost, I give Allah praise for assisting me in completing this work with his strength and direction.

I want to sincerely thank everyone who has supported me throughout the years, shown me kindness and trust, and helped my thesis operate smoothly.

I would want to express my sincere gratitude to my parents, siblings, and other family members for their support and conviction in me. I want to thank my wife in especially for all of her help. Without her, I wouldn't have had the drive to finish this project.

I thank all my teachers that guided me during my educational life. Special thanks go to all my dear friends for their valuable friendship.

I would like to thank my supervisor Professor Adel Aissaoui. This dissertation wouldn't have been possible without his direction, instruction, support, suggestions, and constructive criticism. I would also like to thank him for his consideration, sympathy, and availability over the years, all of which enabled me to advance my scientific knowledge.

I am especially grateful to Mr. Beggas Mohamed, doctor at the University of El Oued, for the honor he has done me to have accepted to judge this work and to chair the defense jury.

I sincerely thank Dr Hariz Bekkar Lourabi, Pr Tidjani menacer, Dr Djenaihi youcef and Pr Dalah Mohamed for agreeing to read my work and to be part of the jury of this thesis. Their presence is a great honor. Thank you for your comments, your criticisms, your advice and simply, for your interest in my work.

Mohammed Said Ferhat

Table of Contents

List of Figures

List of Symbols

Let consider the preliminary notations

$\mathbb{N}$	the set of positive integers
$\mathbb{R}$	the set of real numbers, or the real line
l	a positive integer, in applications having its value in $\{1, 2\}$
d	a positive integer, in applications having its value in $\{1, 2, 3\}$
c	a generic positive constant, the value of which may change from place to place
h	the finite element mesh size
k	the time step size
r_+	the positive part of r =$\max\{r, 0\}$
diam (A)	diameter of the set A
ψ_A	the indicator function of the set A
A^h	the finite element space for the set A
$\forall$	for all
δ_{ij}	the Kronecker delta
a.e.	almost everywhere
e.g.	for example
i.e.	that is
$[0, T]$	the time interval of interest, $T > 0$
δw_n	the backward divided difference = $(w_n - w_{n-1})/k$
Π^h	the finite element interpolation operator
Λ^m	the mth power of the operator Λ
$\mathbb{R}^d$	the d-dimensional Euclidean space
$\mathbb{S}^d$	the space of second-order symmetric tensors on $\mathbb{R}^d$

Let Ω be an open, bounded, and connected set in $\mathbb{R}^d$, we denote by

$\overline{\Omega}$	the closure of Ω
Γ	the boundary of $\Omega : \Gamma = \partial\Omega$
ν	the unit outward normal on the boundary Γ
$v_\nu, \boldsymbol{v}_\tau$	the normal and tangential component of vector field $\boldsymbol{v}$
Γ_i	the parts of the boundary Γ, $(i = 1, 2, 3)$
$L^p(\Omega)$	the Lebesgue space of p-integrable functions on Ω, with the usual modification if $p = \infty$
$W^{q,p}(\Omega)$	the Sobolev space of functions whose weak derivatives of orders q or less are p-integrable on Ω $\quad (q \in \mathbb{N}$ and $p \in \{1. + \infty\})$
$H^q(\Omega)$	the Sobolev space $W^{q,2}(\Omega)$
$H^1(\Omega)$	the first-order Sobolev space
H	the set $\{\mathbf{u} = (u_i) \, : \, u_i \in L^2(\Omega), \, 1 \leq i \leq d\} = L^2(\Omega)^d$
H_1	the set $\{\mathbf{u} \in H \,\vert\, \varepsilon(\mathbf{u}) \in \mathcal{H}\} = \mathrm{H}^1(\Omega)^d$
$\mathcal{H}$	the set $\{\boldsymbol{\tau} = (\tau_{ij}) \,\vert\, \tau_{ij} = \tau_{ji} \in L^2(\Omega), \, 1 \leq i, j \leq d\} = L^2(\Omega)_s^{d\times d}$
$\mathcal{H}_1$	the set $\{\boldsymbol{\tau} \in \mathcal{H} \,\vert\, \tau_{ij,j} \in H\}$
$\mathrm{H}^{\frac{1}{2}}(\Gamma)$	the Sobolev space of order $\frac{1}{2}$ on Γ
$\mathrm{H}^{-\frac{1}{2}}(\Gamma)$	the space dual of $\mathrm{H}^{\frac{1}{2}}(\Gamma)$

Let X be a real Hilbert space and $d \in \mathbb{N}^*$, we denote by

X^d	the set $\{x = (x_i) \, : \, x_i \in X, \, 1 \leq i \leq d\}$
$X_s^{d\times d}$	$\{x = (x_{ij}) \, : \, x_{ij} = x_{ji} \in X, \, 1 \leq i, j \leq d\}$
$(. \, , \, .)_X$	the scalar product of X
$\vert \, . \, \vert_X$	the norm of X
$\mathcal{L}(X)$	the space of linear and continuous maps in X
$C(\mathbb{R}, X)$	the space of continuous functions from $\mathbb{R}$ in X
$C^1(\mathbb{R}, X)$	the space of continuously differentiable functions from $\mathbb{R}$ in X

For a function ψ, we denote by

$\dot{\psi}, \ddot{\psi}$	the first and second derivatives of ψ with respect to time
$\partial_i \psi$	the partial derivative of ψ with respect to the ith component x_i
$\varepsilon(\psi)$	linearized or small deformations operator, i.e., $\varepsilon(\psi) = (\varepsilon_{ij}(\psi))$, $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$
Divψ	the divergence operator of ψ, i.e., Div$(\psi) = (\psi_{ij,j})$
$\nabla\psi$	the gradient operator of f, i.e., $\nabla\psi = (\partial_1\psi, ...\partial_d\psi)$
$\partial\psi$	the subdifferential of the function ψ

General Introduction

Problems of contact, with or without friction, between two deformable materials or between a deformable material and a rigid foundation abound in industry and in everyday life. Mechanical linkages, braking systems, combustion engines and metalworking are usual examples. Because of this importance of these physical phenomena, considerable efforts have been devoted to the study of these contact problems.Their modelling and mathematical analysis, including existence and uniqueness results, was developed in a large number of works, see for instance [55, 35, 52, 56], and the references therein. Their numerical analysis, including error estimate for discrete scheme and numerical simulations, can be found in [36, 39, 42]

The selection of the contact boundary conditions is a significant issue in the modeling of the contact phenomena, which is still a subject of research. The so-called Signorini condition is one of the most often used boundary conditions, appearing in both engineering and mathematics literature. It was introduced in [53], describes the contact with a perfectly rigid foundation and is expressed in terms of unilateral constraints for the displacement field. The normal compliance contact condition was introduced in [13] and used in a large number of papers, see [35, 40, 52] and the references therein. It represents a regularization of the Signorini contact condition and it describes the contact with an elastic foundation. It should be noted that the contact conditions, which are employed in the majority of the related works on the topic, are defined in terms of the displacement field. However, it seems more suitable to characterize typical compliance conditions in terms of the normal velocity when the contact surfaces are lubricated. These settings, also known as normal damped response conditions, have been utilized in a number of articles (see, e.g., [6, 14, 26, 49]) and the references therein).

Numerous publications have employed constitutive laws, which use internal variables to characterize how a material changes as a result of deformation. Some of the internal state variables considered by many authors are the work-hardening of materials, the absolute temperature, the adhesion field and the damage field. See for examples [3, 14, 33].

The damage is a crucial subject in design engineering since it directly impacts the structure or component's usable life. On it, there is a large quantity of engineering literature. The virtual power idea and thermodynamical considerations were used to create general models of mechanical damage in [31]. Contact problems using viscoelastic and viscoplastic materials, involving the damage's effect, were explored in [2, 13, 33].

The adhesion is important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason, adhesive contact between bodies, when a glue is added to prevent the surfaces from relative motion, has recently received increased attention in the literature. Basic modelling can be found in [28, 29, 30]. Analysis of models for adhesive contact can be found in [15, 16, 19] and in the monographs [52, 54].

The piezoelectric effect represents the coupling between the mechanical and the electrical proprieties of the material, it can be considered as an interaction between two phenomena: the direct piezoelectric effect (a mechanical deformation generates an electric field in the material) and the inverse piezoelectric effect (the generation of stress when an electric field is applied). The piezoelectricity is very useful within many applications that involve the production and detection of sound, generation of high voltages, electronic frequency generation, microbalances, and ultra fine focusing of optical assemblies. It is also the basis of a number of scientific instrumental techniques with atomic resolution, such as scanning probe microscopes. The piezoelectric effect also has its use in biomechanics, bio-medicine and structural mechanics.

During the last years, there is a considerable mathematical interest in contact problems involving piezoelectric phenomena (See, e.g., [8, 10, 37]). There are general models for electro-elastic materials with piezoelectric effects in [46, 58, 59] and the references therein. Piezoelectric contact problems arising viscoplasticity have been studied in [2, 45]. Both quasistatic or dynamic contact problems for viscoelastic piezoelectric materials have been taken into account in [60, 51, 43, 57].

The thesis is broken up into three chapters. After reviewing the formalization of the laws of behavior, boundary conditions, and mechanical and electrical phenomena in the first chapter, we offer the physical settings and mathematical models that will be the focus of our investigation in the following two chapters. Preliminaries in mathematics are also presented. In the second chapter, we discuss solutions to the quasistatic frictional contact problem that arises when an elastic-viscoplastic body comes into contact with an obstacle. Normal damped response and a local friction law are used to represent the contact. With two internal variables that may describe a temperature parameter and the damage to the contacting surface, the elastic-viscoplastic material can also characterize both. We present a variational formulation of the problem and demonstrate that the model has a single weak solution. This part contains original results published in collaboration with my supervisor Professor Adel Aissaoui ([25]). In the same way this chapter contains the case of the contact between two viscoelastic piezoelectric bodies with adhesion and damage. The third chapter of this thesis is devoted to the resolution of friconless contact between a visco-elastic piezoelectric body with long memory and a foundation is studied. The contact is modelled by a normal compliance. We present a variational formulation of the problem and prove the existence and uniqueness of the weak solution. A fully discrete scheme for the problem is proposed, and error estimates on the numerical solutions is derived.

Chapter 1

Modeling and Mathematical Background

Contents

1.1 Introduction

This chapter represents a brief reminder of the continuum mechanics where we will introduce the general physical setting of contact problems to be studied in this thesis, the constitutive laws of behavior of the different materials, the contact boundary conditions . Then we present some mathematical preliminaries which will be used subsequently. We begin by reviewing a few conclusions relating to functional spaces, variational inequalities, evolution equations, Gronwall's lemma, and a few more theorems that will be regularly used to prove existence and uniqueness results for the contact problems. Finally, we recall some generalities on the finite element method. The bibliographical references will be specified later in each of the following paragraphs.

1.2 Contact Problem Modeling

1.2.1 Physical Setting

This section is devoted to presenting the general physical setting investigated in this thesis as well as the mathematical formulations useful for the study of contact problems. The contact phenomena considered in this thesis are described by the following physical settings:

Physical Setting n. 1: (Thermo-Mechanical Problem)

We consider a material body which occupies a bounded domain Ω with a regular surface Γ, and let $\Gamma_1, \Gamma_2, \Gamma_3$ be a partition of Γ into three disjoint measurable parts such that $meas\,(\Gamma_1) > 0$. Let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Assuming that the body is fixed on $\Gamma_1 \times (0, T)$ the displacement field is shown to vanish there. A volume force of density f_0 acts on $\Omega \times (0, T)$ and surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$. We admit the possibility of an external heat source applied in $\Omega \times (0, T)$, given by the function q (see figure 1.1).

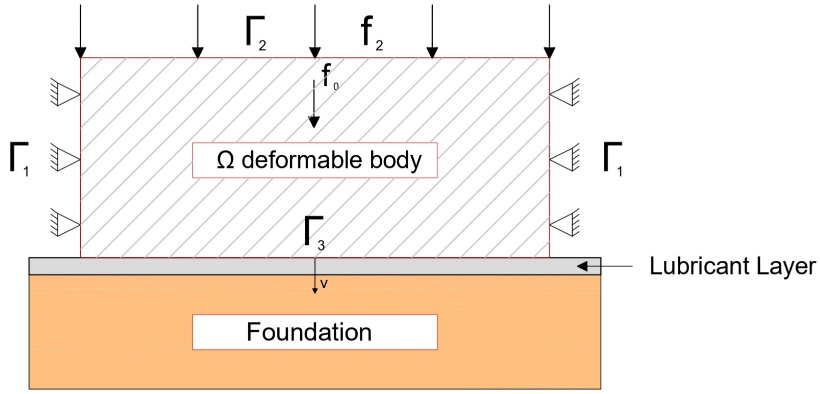


Figure 1.1: Physical setting 1

We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. This leads to a quasistatic approach of the process. In furthermore, the body comes into contact with a reactive foundation over the potential contact surface Γ_3.

We take into account both the mechanical and thermal properties of the material body. Our objective will be to study the evolution of these properties during the time interval $[0, T]$, $T > 0$, under the hypothesis of small transformations.

We will use this physical setting in the second chapter of this thesis.

Physical Setting n. 2: (Electro-Mechanical Problem between two bodies)

Let us examine two material bodies that are located in the space $\mathbb{R}^d$ $(d = 2, 3)$ and have two bounded domains Ω^1, Ω^2. To show that the quantity is related to the domain Ω^l, we use the superscript l. The superscript l in the following ranges from 1 to 2. The boundary Γ^l of each domain Ω^l is divided into two measurable parts Γ_a^l and Γ_b^l, and three disjoint measurable parts Γ_1^l, Γ_2^l and Γ_3^l, under the assumption that the boundary is Lipschitz continuous, taking into account that $meas\left(\Gamma_1^l\right) > 0$ and $meas\left(\Gamma_a^l\right) > 0$. Forces f_0^l and volume electric charges of density q_0^l are applied to the body Ω^l. The displacement field vanishes at Γ_1^l since it is assumed that the bodies are clamped. The part Γ_2^l is affected by the surface tractions f_2^l. A surface electric charge with a density of q_2^l is established on Γ_b^l the electrical potential vanishes on Γ_a^l. Along their common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$, the two bodies are in contact (see figure 1.2).

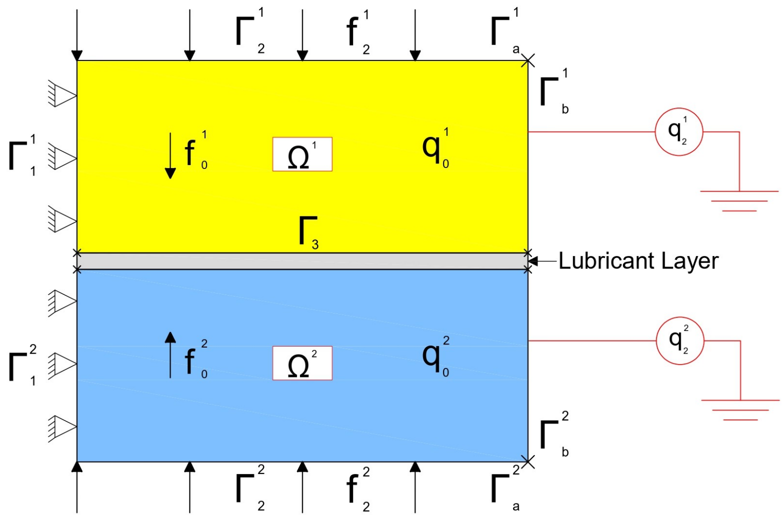


Figure 1.2: Physical setting 2

The difference from the previous physical framework results from the fact that now we consider the mechanical properties and also the electrical properties of the material body. We are interested to study the evolution of these properties in the time interval $[0, T]$, $T > 0$, in the hypothesis of small transformations, by admitting that the process is quasistatic. We will also use this physical setting in the second chapter of this thesis.

Physical Setting n. 3: (Electro-Mechanical Problem with numerical simulations)

We consider an electro-viscoelastic body which occupies a bounded domain Ω with a regular surface Γ, and let Γ_1, Γ_2, Γ_3 be a partition of Γ into three disjoint measurable parts on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Assuming that the body is fixed on $\Gamma_1 \times (0, T)$ the displacement field is shown to vanish there. A volume force of density f_0 acts on $\Omega \times (0, T)$ and surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_2 \times (0, T)$. On Γ_3 the potential contact surface, the body is in contact with a foundation (see Figure 1.3).

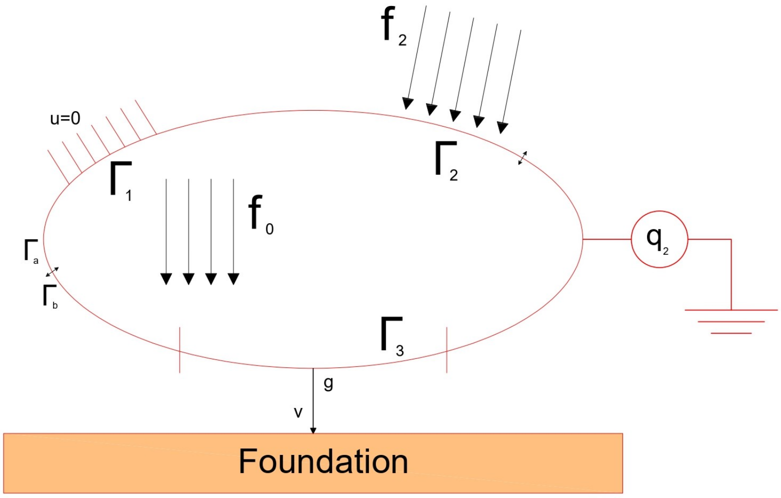


Figure 1.3: Physical setting 3

Before the description of the mathematical models associated with the physical frameworks presented above, we give some notations and conventions that we will use throughout this thesis. $\mathbb{R}^d$ represents the real linear space of dimension d, where d is typically one, two, or three. The linear space of second-order symmetric tensors on $\mathbb{R}^d$, also known as the space of symmetric matrices of order d, is denoted by the symbol $\mathbb{S}^d$. The canonical inner products and the associated $\mathbb{R}^d$ and $\mathbb{S}^d$ norms are

$$\mathbf{u}.\mathbf{v} = u_i v_i, \quad |\mathbf{u}| = (\mathbf{u}.\mathbf{u})^{\frac{1}{2}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d,$$

respectively. The index that comes after a comma denotes the partial derivative with regard to the associated component of the independent spatial variable, and the indices i and j run from 1 to d throughout this thesis. The summation convention over repeated indices is also used. We then employ the usual notation for Lebesgue and Sobolev spaces related to Ω and Γ.

We denote by $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$ the displacement field, $\boldsymbol{\sigma} : \Omega \times [0,T] \to S^d$ the stress field, $\varphi : \Omega \times [0,T] \to \mathbb{S}^d$ the electric potential field, $\boldsymbol{D} : \Omega^l \times [0,T] \to \mathbb{R}^d$ the electric displacement field, $\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and $E(\varphi) = -\nabla\varphi$ the field of linearized deformations and the electric field.

We denote by v_ν and $\boldsymbol{v}_\tau$, respectively, the normal component and the tangential part of any vector field $\boldsymbol{v}$ defined on Γ, given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{1.2.1}$$

In the same way, the stress tensor's normal component and tangential part, which are both designated by σ_ν and $\boldsymbol{\sigma}_\tau$, are also provided by

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{1.2.2}$$

We note that , $\boldsymbol{\sigma}_\tau$ is a tangent vector to Γ while σ_ν is a scalar We denote partial derivatives and components by subscripts, e.g., the components of the linearized strain tensor $\varepsilon(\mathbf{u})$ are given by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.2.3}$$

Later in this thesis, we will use $\nabla\varphi$ to represent the gradient of φ for a given scalar function φ, i.e. $\nabla\varphi = (\varphi_{,i})$.

Equations of Motion and of Equilibrium

We begin with the mathematical model which explains how the body evolves within the physical framework. In the general case, the evolution of a material body is described by the following Cauchy equation of motion:

$$\mathrm{Div}\, \boldsymbol{\sigma} + f_0 = \rho \ddot{\boldsymbol{u}} \quad \text{in } \Omega \times (0, T), \tag{1.2.4}$$

where f_0 represents the density of applied forces, and ρ is the mass density. The divergence operator " Div" is defined as $\mathrm{Div}(\boldsymbol{\sigma}) = (\sigma_{ij,j})$. We eliminate the inertial factors from the equations of motion to obtain the following equilibrium equations when the external forces and tractions vary slowly over time and the system's accelerations are negligible.

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \quad \text{in } \Omega \times (0, T). \tag{1.2.5}$$

Dynamic processes are those modeled by the equations of motion (1.2.4), while quasistatic processes are those modeled by the equilibrium equations (1.2.5).

In the case of a piezoelectric material, there are new unknowns represented by

The electric potential field $\varphi : \Omega^l \times [0, T] \to \mathbb{S}^d$, and the electric displacement $\boldsymbol{D} : \Omega \times [0, T] \to \mathbb{R}^d$ are new unknowns in the case of a piezoelectric material, necessitating the use of another equilibrium equation to address the issue. The Maxwell-Gauss equation or the charge conservation equation describes the evolution of a body in this state.

$$\operatorname{div} \boldsymbol{D} = q_0 \quad \text{in } \Omega \times (0, T), \tag{1.2.6}$$

where q_0 is the density of volumetric electric charges subjected to the body Ω, and $\operatorname{div} \boldsymbol{D} = (D_{i,i})$.

Currently, we have more unknown functions than equations, thus the description of our models is not complete. For instance, in the case d = 3, there are fifteen unknowns functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \varepsilon)$, but only three equations in (1.2.4) or (1.2.5) and six relations in (1.2.3). Physical factors also indicate that the problem's description until now is incomplete. Consequently, The preceding equations are therefore insufficient, on their own, to describe the equilibrium of material bodies. It must then be supplemented by other relations which characterize the behavior of each type of material.

1.2.2 Behavior Laws

The constitutive law, the balance equation, the boundary conditions, the contact conditions, and the initial conditions must all be discussed in order to offer a mathematical model for a particular contact process. The characteristics of each type of material are described by their rules of behavior. Although they must adhere to a few fundamental axioms and invariance rules, their origin is frequently experimental. It is a whole series of tests that must be imagined and carried out to establish a law of behavior, We refer the reader to [35] for a basic overview of numerous diagnostic experiments that offer data required for developing constitutive laws for particular materials and relaxation tests. We cite by way of example four classic examples of tests on solids: monotonic loading tests, load-discharge tests, creep tests.

The relationship between the stress tensor $\boldsymbol{\sigma}$, the strain tensor ε and their derivatives is represented by the law of behavior (also known as the constitutive law) in the description of purely mechanical processes.

This definition changes slightly in the description of electro-mechanical phenomena, because here we must also take into consideration the electric displacement field $\boldsymbol{D} = D_i$ as well as the electric field $E(\varphi) = -\nabla\varphi$ We then present the laws of behavior involved in this thesis.

Law of Behavior of Elastic Materials

Elasticity is the capacity of a body to withstand a deforming force and to regain its original size and shape when the force has been eliminated. Viscosity, on the other hand, is a gauge of a fluid's resistance to flow. A fluid that has a high viscosity opposes movement. Low viscosity fluids move freely. As an illustration, water flows better than syrup because it is less viscous. Gels, syrups, and honey are examples of high viscosity materials.

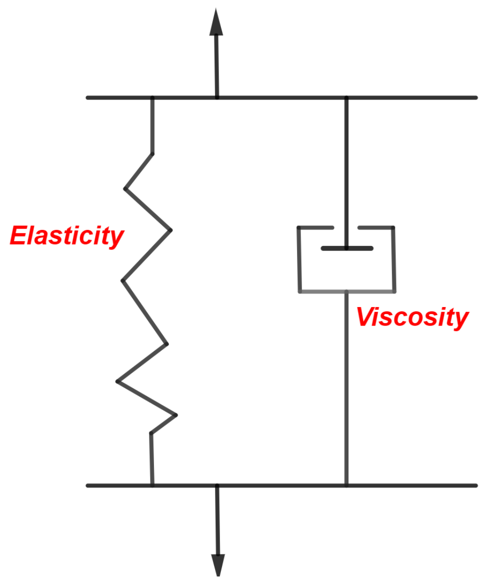


Figure 1.4: Kelvin-Voigt model

Viscoelasticity is a property of materials that, when deformed, exhibit both viscous and elastic properties. Significant viscoelastic effects can be seen in synthetic polymers, wood, human tissue, metals at high temperatures..

A general elastic constitutive law is given by

$$\boldsymbol{\sigma} = \mathcal{M}\varepsilon(\boldsymbol{u}), \tag{1.2.7}$$

where $\mathcal{M}$ is the elasticity operator, assumed to be nonlinear. In particular, if $\mathcal{M}$ is a linear operator, (1.2.7) leads to the constitutive law of linearly elastic materials: $\sigma_{ij} = m_{ijkl}\varepsilon_{kl}(\boldsymbol{u})$, where σ_{ij} are the components of the stress tensor σ and m_{ijkl} are the components of the elasticity tensor $\mathcal{M}$. Let $d = 3$, when the material is linear and isotropic, the elasticity tensor is characterized by only two positive coefficients. Thus, the constitutive law of a linearly elastic isotropic material is given by

$$\boldsymbol{\sigma} = 2\gamma\boldsymbol{\varepsilon}(\boldsymbol{u}) + \delta(tr\boldsymbol{\varepsilon}(\boldsymbol{u}))I_3$$

Here γ and δ are the Lam$\acute{e}$ coefficients, I_3 represents the identity tensor on $\mathbb{R}^3$ and $tr\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the trace of the tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$, $tr\boldsymbol{\varepsilon}(\boldsymbol{u}) = \varepsilon_{ii}(\boldsymbol{u})$,

In components, we have

$$\sigma_{ij} = 2\,\gamma\,\varepsilon_{ij}(\boldsymbol{u}) + \delta\,\varepsilon_{kk}(\boldsymbol{u})\delta_{ij}, \tag{1.2.8}$$

where δ_{ij} are the components of the unit matrix I_3.
By inverting the linear elastic law (1.2.8), we get

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij}$$

where the Poisson's ratio is represented by ν , and E is Young's modulus. The following table represents some materials with their coefficients (see [22, p.151])

Material	Young's modulus (GPa)	Poisson' ratio
Aluminium alloys	69–73	0.33
Bronze	96–120	0.34
Cast iron	110–168	0.25–0.29
Copper alloys	110–115	0.33–0.38
Magnesium alloys	45	0.35
Steels	193–207	0.25–0.29
Titanium alloys	114	0.34
Tungsten	340–380	0.2

Table 1.1: Young's modulus and Poisson' ratio for some materials

Law of Behavior of Viscoelastic Materials

The investigation of the mechanical properties of materials such as pastes or waxes, has highlighted the insufficiency of the theory of elasticity. Indeed, some phenomena, such as creep or relaxation cannot be described by elastic behavior laws. This is why the viscoelastic and viscoplastic models were introduced.

a) Short-Memory Viscoelastic Constitutive Law

The following formula offers a general short-memory viscoelastic constitutive law

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{B}\varepsilon(\boldsymbol{u}), \tag{1.2.9}$$

In linearized viscoelasticity (1.2.9) leads to the Kelvin-Voigt relation

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl}(\dot{\boldsymbol{u}}) + b_{ijkl}\varepsilon_{kl}(\boldsymbol{u}),$$

where the symbols σ_{ij}, a_{ijkl} and b_{ijkl} stand for the stress tensor $\boldsymbol{\sigma}$, the viscosity tensor $\mathcal{A}$ and the elasticity tensor $\mathcal{B}$, respectively.

The constitutive law of a homogeneous viscoelastic isotropic material is given by

$$\sigma_{ij} = \big(a_1\varepsilon_{kk}(\dot{\boldsymbol{u}}) + \lambda_1\varepsilon_{kk}(\boldsymbol{u})\big)\delta_{ij} + 2\big(a_2\varepsilon_{ij}(\dot{\boldsymbol{u}}) + \lambda_2\varepsilon_{ij}(\boldsymbol{u})\big),$$

where λ_1, λ_2 represent the two Lam$\acute{e}$ coefficients, and a_1, a_2 represent the two viscosity coefficients. (the four constants all have positive values.)

If we consider the effect of damage to the material during contact, we obtain a generalization of the prior law, which is the constitutive law of viscoelasticity, with damage having the form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}(\varepsilon(\boldsymbol{u}(t)), \alpha(t)), \quad \text{in } \Omega \times (0, T) \tag{1.2.10}$$

The differential inclusion below describes the evolution of the damage field represented by the function α

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni \phi\big(\varepsilon(\mathbf{u}), \alpha\big),$$

where *K* is the set of permitted damage functions described by

$$K = \{\xi \in V : 0 \leq \xi(x) \leq 1 \text{ a.e. } x \in \Omega\},$$

k represents the damage diffusion constant, assumed positive, φ_K is the indicator function of the set *K* and $\partial\varphi_K$ represents its subdifferential. ϕ is a given constitutive function which describes the sources of the damage in the system which results from tension or compression.

b) Long-Memory Viscoelastic Constitutive Law

The following formula offers a general long-memory viscoelastic constitutive law

$$\boldsymbol{\sigma}(t) = \mathcal{F}\varepsilon\left(\boldsymbol{u}(t)\right) + \int_0^t \mathcal{R}\left(t - s, \varepsilon\left(\boldsymbol{u}(s)\right)\right) ds, \tag{1.2.11}$$

We allow the location of the point to affect both the relaxation operator $\mathcal{R}$ and the elasticity operator $\mathcal{F}$. The operator $\mathcal{R}$ is also time-dependent, as shown in (1.2.11).

In the linearized viscoelasticity, (1.2.11) leads to the relation

$$\sigma_{ij}(t) = f_{ijkl}(\boldsymbol{u}(t)) + \int_0^t r_{ijkl}(t-s), \varepsilon_{kl}(\boldsymbol{u}(s))\text{ds},$$

Law of Behavior of Elastic-Thermo-Viscoplastic Materials with Damage

In this case the constitutive law is given by

$$\boldsymbol{\sigma}(t) = \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}\big(\varepsilon(\mathbf{u}(t))\big) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big)ds,$$

where θstands for temperature, β for damage field, $\mathcal{A}$ and $\mathcal{B}$ for purely viscous and elastic properties of the material, respectively, and $\mathcal{G}$ is a nonlinear constitutive function that explains the material's viscoplastic behavior.

The heat equation, derived from the principle of energy conservation, governs the evolution of the temperature field. It is defined by the following differential equation

$$\dot{\theta} - k_0\Delta\theta = \psi\big(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta, \beta\big) + q,$$

where q is a known volume heat source, and ψ is a nonlinear constitutive function that describes the heat generated by internal forces.

Law of Behavior of Electro-Viscoelastic Materials with Damage

A general electro-viscoelastic constitutive law may be written as

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{B}\varepsilon(\mathbf{u}) - (\mathcal{E})^*E(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{1.2.12}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\mathbf{u}) + \mathcal{C}E(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{1.2.13}$$

where $E(\varphi) = -\nabla\varphi$ stands for the electric field, $\mathcal{E}$ stands for the third order piezoelectric tensor and $\mathcal{E}^*$ represents its transposition; $\mathcal{A}$ and $\mathcal{B}$ stand for the viscosity and the elasticity operators, respectively, and $\mathcal{C}$ denotes the electric permitivity tensor,

By adding the effect of damage to the material during contact, the constitutive law will be the following

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{B}\big(\varepsilon(\mathbf{u}), \alpha\big) - (\mathcal{E})^* E(\varphi) \qquad \text{in } \Omega \times (0, T), \tag{1.2.14}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\mathbf{u}) + \mathcal{C}E(\varphi) \qquad \text{in } \Omega \times (0, T), \tag{1.2.15}$$

1.2.3 Boundary Conditions

Now let's discuss the system's boundary conditions. We suppose that the boundary Γ is Lipschitz. Consequently, the outer unit normal vector is defined at almost every point.

Displacement and Traction Boundary Conditions

We suppose that the body is held fixed on Γ_1 , where the displacement field also vanishes

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1 \times (0, t), \tag{1.2.16}$$

which represents the displacement boundary condition. Additionally, we suppose that the part Γ_2 is subject to surface tractions of density f_2, thus

$$\boldsymbol{\sigma}\nu = f_2, \quad \text{on } \Gamma_2 \times (0, T), \tag{1.2.17}$$

This condition is called the traction boundary condition.

Electrical Boundary Conditions

We suppose that the electrical potential disappears on Γ_a, and a surface electric charge of density q_2 is required on Γ_b. Therefore, the two equations determine these conditions

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0, T), \tag{1.2.18}$$

$$\boldsymbol{D}.\boldsymbol{\nu}^l = q_2 \qquad \text{on } \Gamma_b \times (0, T), \tag{1.2.19}$$

Our main focus will now be on a description of various contact conditions on the contact surface Γ_3, These naturally divide into conditions in the normal direction and those in the tangential directions.

Bilateral Contact Condition

The condition where contact between the body and the foundation is maintained throughout the process is referred to as "bilateral contact". and it is given by

$$u_\nu = 0 \quad \text{on } \Gamma_3 \times (0, t), \tag{1.2.20}$$

This situation can be found in a variety of machines, as well as in moving parts and other mechanical equipment components. It was considered by several authors, for details see [35, 52] and the references therein.

Normal Compliance Condition

The normal compliance condition determines a reactive normal traction or pressure based on the interpenetration of the asperities on the surface of the body and those of the foundation. It can be generally given as follows

$$-\sigma_\nu = p_\nu(u_\nu - g), \tag{1.2.21}$$

where $u_\nu - g$ is a measure of the interpenetration of the asperities and is positive when there is contact, p_ν is a nonnegative specified function that disappears for a negative argument. The normal compliance function p_ν is frequently cited as an example by

$$p_\nu = c_\nu r_+, \tag{1.2.22}$$

where $r_+ = \max\{0, r\}$ stands for the positive part of r and c_ν is a positive constant that indicates the surface stiffness coefficient.

The Signorini contact condition, in which the foundation is assumed to be fully rigid, is an idealization of the normal compliance that is frequently used in engineering literature and can also be found in mathematical publications. Formally, it is derived from the usual compliance condition in the limit of infinite surface stiffness coefficient, when interpenetration is prohibited. This suggests that contact with a rigid support should be thought of as a special case of contact with a deformable support, whose resistance to compression rises. The supplementary form of the Signorini contact condition is as follows.

$$u_\nu \leq g \qquad \sigma_\nu \leq 0 \qquad \sigma_\nu(u_\nu - g) = 0 \tag{1.2.23}$$

Normal Damped Response Condition

The so-called normal damped response condition refers to a lubricated contact (see [49]). This condition presupposes that the normal stress $\boldsymbol{\sigma}_\nu$ on the contact surface is linked to the normal velocity $\dot{\mathbf{u}}_\nu$, that is

$$-\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \tag{1.2.24}$$

where p_ν is a is a specified function. An example is given by taking

$$p_\nu(r) = kr, \tag{1.2.25}$$

with k $\geq$ 0. The friction law adopted in the frictional case is as follows

$$-\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \tag{1.2.26}$$

where $\dot{\mathbf{u}}_\tau$ stands for tangential velocity, p_τ is a prescribed vector-valued function, and $\boldsymbol{\sigma}_\tau$ represents the tangential force acting on the contact boundary.

Frictional Contact Conditions

Now, let's talk about the tangential conditions, generally known as the friction laws. The first is the so-called frictionless situation, where the friction force, or tangential component of the stress, vanishes, i.e.

$$\boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_3 \times (0,t), \tag{1.2.27}$$

The contact is frictional when the friction force $\boldsymbol{\sigma}_\tau$ does not disappear on the contact surface. Then we provide some friction laws, which are represented as relations between the normal and tangential components of the stress field σ and the tangential displacement $\boldsymbol{u}_\tau$ or the tangential velocity $\dot{\boldsymbol{u}}_\tau$.

Coulomb's Friction Law

In most cases, the Coulomb law of dry friction or one of its variations is used to model frictional contact. This law states that the tangential traction $\boldsymbol{\sigma}_\tau$ can approach a bound F_b, also called as the friction bound, which is the maximum amount of frictional resistance that the surfaces can produce and which is reached when a relative slip motion begins.

$$\begin{cases} |\sigma_\tau| \leq \mu|\sigma_\nu|, \\ |\sigma_\tau| < \mu|\sigma_\nu| \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = \mu|\sigma_\nu| \Rightarrow \exists\lambda \geq 0 : -\lambda\sigma_\tau = \dot{u}_\tau, \end{cases} \tag{1.2.28}$$

where $\mu \geq 0$ represents the friction coefficient.

1.2.4 Contact Processes with Adhesion

The adhesion has numerous industrial uses, including the joining of nonmetallic components and the prevention of composite materials from delaminating. To model the adhesion phenomenon, it is necessary to add the adhesion process to the contact description .

We introduce the bonding field, a surface internal variable that is denoted by β and describes the fractional density of active bonds on the contact surface. If $\beta = 0$, all the bonds are severed and inactive, and there is no adhesion; if $\beta = 1$, the adhesion is total adhesion and all the bonds are active; and if $0 < \beta < 1$, there is only partial adhesion. [1,3,10,19,21,23,24] and the references therein contain general models with adhesion.

An ordinary differential equation that considers the contact surface and depends on displacement governs the bonding field's evolution. For more details concerning the modeling of the adhesive contact, we refer to the books [30, 54].

If the compressive part of the contact stress is assumed to be represented by the normal compliance and a vanishing gap function, then the normal compliance contact condition with adhesion is given by

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu\beta^2(-R(u_\nu))_+. \tag{1.2.29}$$

Here γ_ν is a positive adhesion coefficient, p_ν is the normal compliance function, and R denotes the truncation operator given by

$$R(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad -L \leq s \leq 0, \\ L & \text{if} \quad s > 0, \end{cases} \tag{1.2.30}$$

where $L > 0$ is the bond's characteristic length, beyond which it offers no further traction (see, e.g., [1, 54]). Thus

$$\tilde{R}(s) = (-R(u_\nu))_+(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad |s| \leq L, \\ 0 & \text{if} \quad s > L, \end{cases} \tag{1.2.31}$$

In Figure 1.5 the graph of the function $\tilde{R}(u_\nu)$ is depicted.

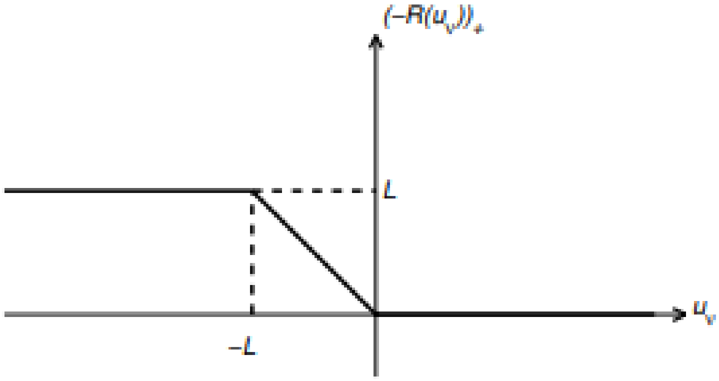


Figure 1.5: The function $\tilde{R}(u_\nu)$

The introduction of the truncation operator R is primarily driven by mathematical considerations, but it is also connected to the finding that for some glues, when the extension is greater than R, the glue stretches plastically without providing further tensile traction. But if we choose L to be really large, we can get back to the situation where the traction is linear in the extension. Thus, the adhesive normal traction is $\gamma_\nu \beta^2(-R(u_\nu))_+$; it is tensile and proportional, with proportionality coefficient γ_ν, to the square of the adhesion field, and to the normal displacement, as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$.

This following differential equation describes the evolution of the adhesion field.

$$\dot{\beta} = \mathbf{H}_{ad}(\beta, \xi_\beta, R_\nu(|u_\nu|), \mathbf{R}_\tau(|\boldsymbol{u}_\tau|)) \qquad \text{on} \quad \Gamma_3 \times [0, T], \tag{1.2.32}$$

$\mathbf{H}_{ad}$ is a general function that disappears when its first variable does too. Such a function was presented as an example in [15, 38].

It was supposed that in addition to β and $R(u_{nu})$, the adhesion rate function also depended on the history of the bonding process, which is represented by

$$\xi_\beta(x, t) = \int_0^t \beta(x, s)ds \qquad \text{on} \quad \Gamma_3 \times [0, T], \tag{1.2.33}$$

For example, we can consider $\mathbf{H}_{ad}(\beta, r) = -\gamma_\nu \beta_+ r^2$, where γ_ν denotes the normal rate coefficient. Let's take another example, in which $\mathbf{H}_{ad}$ depends on its three variables

$$\mathbf{H}_{ad}(\beta, \xi_\beta, r) = -\gamma_1 \beta_+ r^2 + \gamma_2 \frac{\beta_+(1-\beta)_+}{1+\xi_\beta^2},$$

where the debonding and rebonding rate factors γ_1, γ_2 are assumed to be positive constants.

Let us now turn to the description of the mathematical models associated with the physical settings above.

1.2.5 Mathematical Formulation of Contact Problems

A system of partial differential equations involving the equation of motion is used to mathematically model the evolution of deformable bodies under the influence of external forces, the law of behavior of materials as well as the boundary conditions, to which they are subject. More specifically, the mechanical problems that we will study are the following

Problem 1: Thermo-Mechanical Problem

Determine the stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$, the displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ and the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}\big(\varepsilon(\mathbf{u}(t))\big) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big) ds, \quad \text{in } \Omega \times (0,T), \tag{1.2.34}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{1.2.35}$$

$$\dot{\theta} - k_0 \Delta\theta = \psi\big(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta, \beta\big) + q, \quad \text{in } \Omega \times (0,T), \tag{1.2.36}$$

$$\dot{\beta} - k_1 \Delta\beta + \partial\varphi_K(\beta) \ni \phi\big(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \theta, \beta\big), \quad \text{in } \Omega \times (0,T), \tag{1.2.37}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{1.2.38}$$

$$\boldsymbol{\sigma}\nu = f_2, \quad \text{on } \Gamma_2 \times (0,T), \tag{1.2.39}$$

$$-\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \quad -\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \quad \text{on } \Gamma_3 \times (0,T), \tag{1.2.40}$$

$$k_0 \frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.2.41}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \quad \text{on } \Gamma \times (0,T), \tag{1.2.42}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0, \qquad \theta(0) = \theta_0, \quad \text{in } \Omega. \tag{1.2.43}$$

The quasistatic evolution of damage in thermoelastic-viscoplastic materials is represented by this problem.

Problem 2: Electro-Mechanical Problem

For l=1,2, determine a stress field $\boldsymbol{\sigma}^l : \Omega^l \times [0,T] \to \mathbb{S}^d$, a displacement field $\mathbf{u}^l : \Omega^l \times [0,T] \to \mathbb{R}^d$, an electric displacement $\boldsymbol{D}^l : \Omega^l \times [0,T] \to \mathbb{R}^d$, an electric potential field $\varphi^l : \Omega^l \times [0,T] \to$

$\mathbb{S}^d$, a damage field $\alpha^l : \Omega^l \times [0,T] \to \mathbb{R}$ and a bounding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}^l = \mathcal{A}^l \varepsilon(\dot{\mathbf{u}}^l) + \mathcal{B}^l\big(\varepsilon(\mathbf{u}^l), \alpha^l\big) - (\mathcal{E}^l)^* E(\varphi^l) \qquad \text{in } \Omega^l \times (0,T), \tag{1.2.44}$$

$$\boldsymbol{D}^l = \mathcal{E}^l \varepsilon(\mathbf{u}^l) + \mathcal{C}^l E(\varphi^l) \qquad \text{in } \Omega^l \times (0,T), \tag{1.2.45}$$

$$\operatorname{Div} \boldsymbol{\sigma}^l + \boldsymbol{f}_0^l = 0 \qquad \text{in } \Omega^l \times (0,T), \tag{1.2.46}$$

$$\operatorname{div} \boldsymbol{D}^l - q_0^l = 0 \qquad \text{in } \Omega^l \times (0,T), \tag{1.2.47}$$

$$\dot{\alpha}^l - \kappa^l \Delta \alpha^l + \partial \varphi_{\mathcal{K}^l}(\alpha^l) \ni \phi^l\big(\varepsilon(\mathbf{u}^l), \alpha^l\big) \qquad \text{in } \Omega^l \times (0,T), \tag{1.2.48}$$

$$\mathbf{u}^l = 0 \quad \text{on } \Gamma_1^l \times (0,T), \qquad \boldsymbol{\sigma}^l \boldsymbol{\nu}^l = \boldsymbol{f}_2^l \qquad \text{on } \Gamma_2^l \times (0,T), \tag{1.2.49}$$

$$\varphi^l = 0 \quad \text{on } \Gamma_a^l \times (0,T), \qquad \boldsymbol{D}^l.\boldsymbol{\nu}^l = q_2^l \qquad \text{on } \Gamma_b^l \times (0,T), \tag{1.2.50}$$

$$\sigma_\nu^1 = \sigma_\nu^1 \equiv \sigma_\nu, \text{ where } -\sigma_\nu = p_\nu([\dot{u}_\nu]) - \gamma_\nu \beta^2 R_\nu([u_\nu]) \qquad \text{on } \Gamma_3 \times (0,T) \tag{1.2.51}$$

$$\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^1 \equiv \boldsymbol{\sigma}_\tau, \text{ where } -\boldsymbol{\sigma}_\tau = p_\tau([\dot{\mathbf{u}}_\tau]) + q_\tau(\beta) \boldsymbol{R}_\tau([\mathbf{u}_\tau]) \qquad \text{on } \Gamma_3 \times (0,T), \tag{1.2.52}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu (R_\nu([u_\nu]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_\tau])\right|^2\big) - \epsilon_a\Big)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{1.2.53}$$

$$\frac{\partial \alpha^l}{\partial \nu} = 0 \qquad \text{on } \Gamma^l \times (0,T), \tag{1.2.54}$$

$$\mathbf{u}^l(0) = \mathbf{u}_0^l, \qquad \alpha^l(0) = \alpha_0^l \qquad \text{in } \Omega^l, \tag{1.2.55}$$

$$\beta(0) = \beta_0 \qquad \text{in } \Gamma_3. \tag{1.2.56}$$

The quasistatic adhesive frictional contact between two electro-viscoelastic bodies with damage and normal damped response is shown in this second problem.

Problem 3: Purely Mechanical Problem

Determine a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$, a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$, an electric displacement field $\boldsymbol{D} : \Omega \times (0,T) \to \mathbb{R}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$

and a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\varepsilon\left(\boldsymbol{u}(t)\right) - \mathcal{E}^{*}E(\varphi) + \int_0^t \mathcal{M}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \alpha(s)\right) ds \quad \text{in } \Omega \times (0,T), \tag{1.2.57}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}E(\varphi) \quad \text{in } \Omega \times (0,T), \tag{1.2.58}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_{\mathcal{K}}(\alpha) \ni \Theta\left(\varepsilon(\boldsymbol{u}), \alpha\right), \quad \text{in } \Omega \times (0,T), \tag{1.2.59}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{1.2.60}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{1.2.61}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{1.2.62}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{1.2.63}$$

$$-\sigma_\nu = p(u_\nu - g), \quad \boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{1.2.64}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.2.65}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{1.2.66}$$

$$\boldsymbol{D}.\nu = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{1.2.67}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \alpha(0) = \alpha_0, \quad \text{in } \Omega. \tag{1.2.68}$$

This third Problem illustrates a frictionless contact between an electro-viscoelastic body and a deformable foundation with long memory and normal compliance.

1.3 Mathematical Preliminaries

This section contains preliminary functional analysis material that will be used subsequently. We begin by reviewing the definitions and properties of several function spaces, including spaces of continuous, continuously differentiable, Sobolev spaces, and spaces of vector-valued functions. All of the function spaces used in this thesis are real. The Banach fixed point theorem, as well as some standard results on variational inequalities and evolution equations, are then recalled and will be used repeatedly in proving existence and uniqueness results for the contact problems. Finally, we present several Gronwall-type inequalities that will be used frequently. This material can be found in many books on functional analysis, e.g., [4, 20].

1.3.1 Function Spaces for Contact Mechanics

Modeling contact problems usually requires the introduction of specific function spaces. We introduce the following spaces using the standard notation for Lebesgue and Sobolev spaces associated with Ω and Γ

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) \,|\, u_i \in L^2(\Omega)\right\}, \qquad H_1 = \{\mathbf{u} \in H \,|\, \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \left\{\boldsymbol{\tau} = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\right\}, \qquad \mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \,|\, \tau_{ij,j} \in H\}.$$

The canonical scalar products of these real Hilbert spaces are

$$(\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \, dx \quad \forall \mathbf{u}, \mathbf{v} \in H,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$
$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in H_1,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_H, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1,$$

where the operators $\varepsilon : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

We denote by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$ the norms associated with the spaces $H, \mathcal{H}, H_1\, and\, \mathcal{H}_1$ respectively,

We now present the Green's formula as follows

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}))_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathbf{u})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \mathbf{u} \, da \quad \forall \mathbf{u} \in H_1. \tag{1.3.1}$$

We end this paragraph with a fixed point resulting from the Banach contraction principle which will be used in Section 2.2.3 of this thesis, in [41] we can found its proof.

Lemma 1.3.1 *Assuming that X is a Banach space with the norm $|\,.\,|_X$ and $T > 0$, let $\Lambda : L^2(0,T;X) \to L^2(0,T;X)$ be an operator such that*

$$|(\Lambda\eta_1)(t) - (\Lambda\eta_2)(t)|_X^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_X^2 \, ds,$$

for each $\eta_1, \eta_2 \in L^2(0,T;X)$, a.e. $t \in (0,T)$ with a constant $C > 0$. Then Λ has a unique fixed point in $L^2(0,T;X)$, that is, there exists a unique $\eta^* \in L^2(0,T;X)$ such that

$$\Lambda\eta^* = \eta^*.$$

1.3.2 Elements of Nonlinear Analysis

In this paragraph, we recall some elements of nonlinear analysis in Hilbert spaces and standard results on variational inequalities and evolution equations.

Strongly Monotone Operators

We begin here with a brief reminder of the strongly monotone and Lipschitz operators. For this, we will take into account a Hilbert space X with the scalar product $(\cdot,\cdot)_X$ and the related norm $\|\cdot\|_{\mathrm{X}}$.

Definition 1.3.1 *Let the nonlinear operator $\mathrm{A} : \mathrm{X} \to \mathrm{X}$ be given. A is said to be*

1. *monotone if $(Au - Av, u - v)_{\mathrm{X}} \geq 0, \quad \forall u, v \in \mathrm{X}.$*
2. *strictly monotone if $(Au - Av, u - v)_{\mathrm{X}} > 0, \quad \forall u, v \in \mathrm{X},\ u \neq v.$*
3. *strongly monotone if there exists a constant $m > 0$ such that*

$$(Au - Av, u - v)_{\mathrm{X}} \geq m||u - v||_{\mathrm{X}}^2, \quad \forall u, v \in \mathrm{X}.$$

4. *nonexpansive if $||Au - Av||_{\mathrm{X}} \leq ||u - v||_{\mathrm{X}}, \quad \forall u, v \in \mathrm{X}.$*

5. *Lipschitz continuous if there exists $L > 0$ such that*

$$||Au - Av||_{\mathrm{X}} \leq L||u - v||_{\mathrm{X}}, \quad \forall u, v \in \mathrm{X}.$$

Theorem 1.3.1 ***(The Banach Fixed Point Theorem)***

Let K *represent a closed, nonempty subset of a Banach space* $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$. *We suppose that* $\Lambda : \mathrm{K} \to \mathrm{K}$ *is a contraction, i.e. there exists a constant* $c \in]0, 1[$ *such that*

$$\| \Lambda(u) - \Lambda(v) \|_{\mathrm{X}} \leq c \| u - v \|_{\mathrm{X}} \quad \forall u, v \in \mathrm{K}.$$

Then the problem has an unique fixed point in K*, or a unique* $u \in \mathrm{K}$ *such that* $\Lambda(u) = u$*.*

We also require a version of the Banach fixed point theorem, which we will recall in the paragraphs that follow. To accomplish this, we define an operator's powers inductively by the formula $\Lambda^m = \Lambda(\Lambda^{m-1})$, for $m \geq 2$.

Theorem 1.3.2 *Suppose that* K *is a closed nonempty subset of a Banach space* $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$*, assume that for some positive integer* m*,* $\Lambda^m : \mathrm{K} \to \mathrm{K}$ *is a contraction. Then* Λ *has a unique fixed point in* K*.*

1.3.3 Variational Inequalities and Evolution Equations

In the following paragraphs of this subsection, The results for elliptic and parabolic variational inequalities as well as for ordinary differential equations in abstract spaces are reviewed.

Elliptic Variational Inequalities.

Definition 1.3.2 *The bilinear form* $a : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is said to be*

1. *continuous or bounded if there exists a number* $M > 0$ *such that*

$$|a(u, v)| \leq M \| u \|_{\mathrm{X}} \| v \|_{\mathrm{X}}, \quad \forall u, v \in \mathrm{X}.$$

2. *X-elliptic if there is a constant* $m > 0$ *such that*

$$a(u,u) \geq m \parallel u \parallel_{\mathrm{X}}^2, \quad \forall u \in \mathrm{X}.$$

3. *symmetric if* $a(u,v) = a(v,u) \quad \forall u, v \in \mathrm{X}.$

Regarding elliptic variational inequalities, we have the following standard existence and uniqueness results.

Theorem 1.3.3 *Let* K *be a nonempty convex and closed subset of a Hilbert space* X. *Assume that* $l : \mathrm{X} \longrightarrow \mathbb{R}$ *is a linear continuous functional and that* $a : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is a continuous and X-elliptic bilinear form. Then there exists a unique solution to the elliptic variational inequality of the first kind*

$$u \in K, \quad a(u, v-u) \geq l(v-u) \quad \forall\, v \in K. \tag{1.3.2}$$

Theorem 1.3.4 *Let* X *be a Hilbert space. Assume that* $a : \mathrm{X} \times \mathrm{X} \longrightarrow \mathbb{R}$ *is a continuous and X-elliptic bilinear form,* $j : \mathrm{X} \longrightarrow \bar{\mathbb{R}}$ *is proper, convex, and l.s.c. on* X *and that* $l : \mathrm{X} \longrightarrow \mathbb{R}$ *is a linear continuous functional. Then there exists a unique solution to the elliptic variational inequality of the second kind*

$$u \in K, \quad a(u, v-u) + j(v) - j(u) \geq l(v-u) \quad \forall\, v \in K. \tag{1.3.3}$$

The obstacle problem for a stretched membrane is a well-known example of an elliptic variational inequality of the first kind,

$$u \in K, \quad \int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq \int_\Omega f(v-u)\, dx \quad \forall\, v \in K.$$

Here, $\Omega \subset \mathbb{R}^2$ is a simply connected domain with smooth boundary Γ and $f \in L^2(\Omega)$. The equilibrium state of an elastic membrane connected to a closed curve Γ and subject to the action of a vertical force of density f is modeled in this problem. The membrane's vertical displacement, which must lie in K, is the unknown u . Theorem 1.3.3 can be used to demonstrate that the obstacle problem has an unique solution.

In this problem, a linearly elastic cylinder of cross-section $\Omega \subset \mathbb{R}^2$ that is in frictional contact on its lateral surface exhibits antiplane shear deformation. Let $g > 0$ represent the specified friction bound, u represent the unknown displacement field, and$f \in L^2(\Omega)$ denote the axial component of the body forces. The problem is to find $u \in H^1(\Omega)$ such that

$$(u, v-u)_{H^1(\Omega)} + g\int_\Gamma (|v| - |u|)da \geq (f, v-u)_{L^2(\Omega)} \quad \forall v \in H^1(\Omega).$$

Since the form $a(\cdot,\cdot)$ is bilinear, the leading operators associated with Theorems (1.3.3) and (1.3.4) are linear. To extend them to nonlinear operators, we let $A : X \longrightarrow X$ be an operator strongly monotone and Lipschitz continuous on X. Then we have the following results

Theorem 1.3.5 *Let K be a nonempty convex and closed subset of a Hilbert space X. and let $A : X \longrightarrow X$ be an operator strongly monotone and Lipschitz continuous on X. Then, for each $f \in X$ there exists a unique solution to the elliptic variational inequality of the first kind,*

$$u \in K, \quad (Au, v-u)_X \geq (f, v-u)_X \quad \forall\, v \in K.$$

Theorem 1.3.6 *Let X be a Hilbert space. Assume that the operator $A : X \longrightarrow X$ is strongly monotone and Lipschitz continuous on X and that $j : \mathrm{X} \longrightarrow \bar{\mathbb{R}}$ is proper, convex, and l.s.c functional. Then, for each $f \in X$ there exists a unique solution to the elliptic variational inequality of the second kind,*

$$u \in K, \quad (Au, v-u)_X + j(v) - j(u) \geq (f, v-u)_X \quad \forall\, v \in K.$$

Parabolic Variational Inequalities.

Assume that V and H be real Hilbert spaces and that the injection map is continuous and V is dense in H. The own dual of the space H and a subspace of the dual V' of V are used to identify it. We write $V \subset H \subset V'$, and we declare that the above inclusions define a Gelfand triple. We represent the norms on the spaces V, H, and V' by $|\cdot|_V$, $|\cdot|_H$ and $|\cdot|_{V'}$ respectively, and we use $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V. Remember that if

$f \in H$, then

$$(f, v)_{V' \times V} = (f, v)_H \quad \forall v \in V.$$

A standard result for parabolic variational inequalities is given below (see, e.g., [9, p.124]).

Theorem 1.3.7 *Let K be a nonempty closed, convex set of W and let $W \subset H \subset W'$ be a Gelfand triple.Assume that $b : (.,.) : W \times W \to \mathbb{R}$ is a symmetric, continuous, bilinear form that satisfies*

$$b(w, w) + d_0 \left|v\right|_H^2 \geq \kappa \left|v\right|_V^2 \quad \forall w \in W,$$

for some constants κ and d_0. Then, for any $u_0 \in K$ and $g \in L^2(0, T; H)$, there exists a unique function $u \in L^2(0, T; W) \cap H^1(0, T; H)$ that satisfies

$$\begin{aligned}
&u(t) \in K \quad \forall t \in (0, T),\\
&(\dot{u}(t), w - u(t))_{W' \times W} + b\big(u(t), w - u(t)\big) \geq \big(g(t), w - u(t)\big)_H\\
&\forall w \in K, \ a.e. \ t \in (0, T),\\
&u(0) = u_0.
\end{aligned}$$

Ordinary Differential Equations in Abstract Spaces.

We recall in this paragraph two results on the evolution equations

Theorem 1.3.8 *Let $\mathrm{W} \subset \mathrm{H} \subset \mathrm{W}'$ be a Gelfand triplet. Assume that $B : \mathrm{W} \to \mathrm{W}'$ is a hemicontinuous and monotone operator that satisfies*

$$(Bw, w)_{\mathrm{W}' \times \mathrm{W}} \geq \mu |w|_W^2 + \lambda, \quad \forall w \in \mathrm{W}, \tag{1.3.4}$$

$$|Bw|_{\mathrm{W}'} \leq C_1(|w|_W + 1), \quad \forall w \in \mathrm{W}. \tag{1.3.5}$$

for some constant $\mu > 0,\ C_1 > 0$ and $\lambda \in \mathbb{R}$. Then given $u_0 \in \mathrm{H}$ and $g \in \mathrm{L}^2(0, T; \mathrm{W}')$, there exists a unique function u satisfies

$$u \in \mathrm{L}^2(0, T; \mathrm{W}) \cap \mathrm{C}^1(0, T; \mathrm{H}), \quad \dot{u} \in \mathrm{L}^2(0, T; \mathrm{W}'),$$

$$\dot{u}(t) + Bu(t) = g(t) \, a.e. \, t \in [0, T],$$

$$u(0) = u_0.$$

Theorem 1.3.9 ***(Cauchy-Lipschitz)*** *Assume that* $(\mathrm{X}, |\cdot|_{\mathrm{X}})$ *is a real Banach space. Let* $\mathrm{G}(t, \cdot) : \mathrm{X} \to \mathrm{X}$ *be an operator defined a.e. on* $[0, T]$*, that satisfies*

$$\begin{cases} (a) \text{ there exists } \quad \mathrm{L_G} > 0 \text{ such that} \\ |\mathrm{G}(t, x) - \mathrm{G}(t, y)|_{\mathrm{X}} \leq \mathrm{L_G}|x - y|_{\mathrm{X}} \quad \forall x, y \in \mathrm{X}, \textit{a.e.}\, t \in [0, T], \\ (b) \text{ there exists } \quad p \geq 1 \text{ such that } \quad \mathrm{G}(\cdot, x) \in \mathrm{L}^p(0, T; \mathrm{X}) \quad \forall x \in \mathrm{X}. \end{cases}$$

Then for every $x_0 \in \mathrm{X}$*, there exists a unique function unique* $x \in \mathrm{W}^{1,p}(0, T; \mathrm{X})$ *such that*

$$\dot{x}(t) = \mathrm{G}(t, x(t)), \;\; \textit{a,e.}\, t \in [0, T],$$

$$x(0) = x_0.$$

1.3.4 Elementary Inequalities

In this subsection we recall inequalities that will be employed in later chapters.

Lemma 1.3.2 ***(Cauchy-Schawrtz Inequality)*** *Let* V *be a vector space with inner product* $(.,.)_V$*. Then for all* $u, v \in V$*. we have*

$$|(u, v)|^2 \leq (u, u) \cdot (v, v)$$

Lemma 1.3.3 ***(Friedrichs-Poincare Inequality)*** *Let* dx *stand for the d-dimensional Lebesgue measure on* $\mathbb{R}^d$*, and let* Ω *be an open, bounded, and connected subset of* $\mathbb{R}^d$ *for some* d*. There exists a constant* C_F *such that for any function* f *in the Sobolev space* $H_0^1(\Omega)$

$$\int_\Omega f^2(x) \, dx \leq C_F \int_\Omega |\nabla f(x)|^2 \, dx.$$

In the study of function spaces and partial differential equations, this inequality is important.

Lemma 1.3.4 ***(Gronwall's Inequality)*** *Let* $a \geq 0$. *Assume* m, n *and* ψ *elements of* $C([0,T];\mathbb{R})$ *such that* $m(t) \geq 0,\ n(t) \geq 0 \quad \forall t \in [0,T]$.

(1) If

$$\psi(t) \leq a + \int_0^t m(s)ds + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

then

$$\psi(t) \leq \left(a + \int_0^t m(s)ds\right) exp\left(\int_0^t n(s)ds\right) \quad \forall t \in [0,T].$$

(2) If

$$\psi(t) \leq m(t) + a\int_0^t \psi(s)ds \quad \forall t \in [0,T],$$

then

$$\int_0^t \psi(s)ds \leq e^{aT}\int_0^t m(s)ds \quad \forall t \in [0,T].$$

For the case m = 0, part (1) of this lemma becomes

Corollary 1.3.1 *Let* $a \geq 0$. *Assume* n, ψ *elements of* $C([0,T];\mathbb{R})$ *such that* $n(t) \geq 0 \quad \forall t \in [0,T]$. *If*

$$\psi(t) \leq a + \int_0^t n(s)\psi(s)ds \quad \forall t \in [0,T],$$

then

$$\psi(t) \leq a exp\left(\int_0^t n(s)ds\right), \quad \forall t \in [0,T].$$

Lemma 1.3.5 *Let* $a \geq 0$. *Assume* m, n *and* ψ *elements of* $C([0,T];\mathbb{R})$ *such that* $m(t) \geq 0,\ n(t) \geq 0 \quad \forall t \in [0,T]$. *If*

$$\frac{1}{2}\phi^2(s) \leq \frac{1}{2}a^2 + \int_0^s m(t)\phi(t)dt + \int_0^s n(t)\phi^2(t)dt, \quad \forall s \in [0,T],$$

then

$$|\phi(s)| \leq \left(a + \int_0^s m(t)ds\right) e^{\int_0^s n(t)dt}, \quad \forall s \in [0,T].$$

For the case m = 0, part (1) of this lemma becomes

Corollary 1.3.2 *Let* $a \geq 0$. *Assume* n, ψ *elements of* $C([0,T];\mathbb{R})$ *such that* $n(t) \geq 0 \quad \forall t \in [0,T]$. *If*

$$\frac{1}{2}\phi^2(s) \leq \frac{1}{2}a^2 + \int_0^s m(t)\phi(t)dt, \quad \forall s \in [0,T],$$

then

$$|\phi(s)| \leq a + \int_0^s m(t)dt, \quad \forall s \in [0,T].$$

1.3.5 Generalities on The Finite Element Method

The finite element method is one of the most effective numerical methods for resolving extremely difficult differential equation problems. One of the major advantages of this method is the fact that it offers the possibility of developing a program that can solve, with few modifications, several types of problems, for details see [18, 54]. The problem domain's finite number of points and subdomains are used in this method. The points, also known as the nodes, are where the values of the provided function are held. The so-called finite elements, which are non-overlapping subdomains connected at nodes on their boundaries, retain piecewise and local approximations of the function that are each uniquely defined in terms of the values held at their nodes. The mesh is a group of discretized nodes and elements, and meshing is the process of creating it. Figure 1.6 shows a typical triangular finite element partition of a two-dimensional domain.

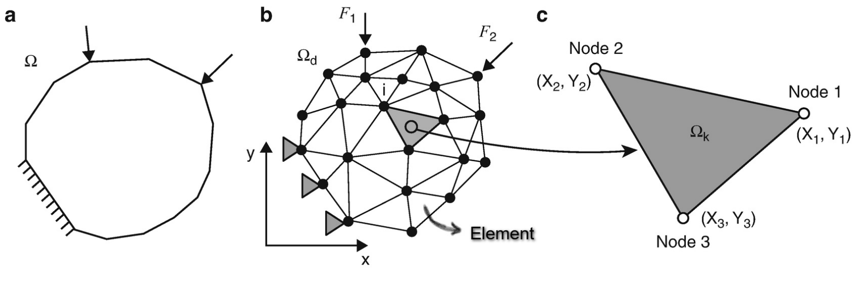


Figure 1.6: Discretization of the domain – triangular elements

The resolution of a physical problem by finite elements follows the following steps (See Figure 1.7), where K stands for the global stiffness matrix, F stands for the general stress vector and U represents the global vector of the problem's nodal variables

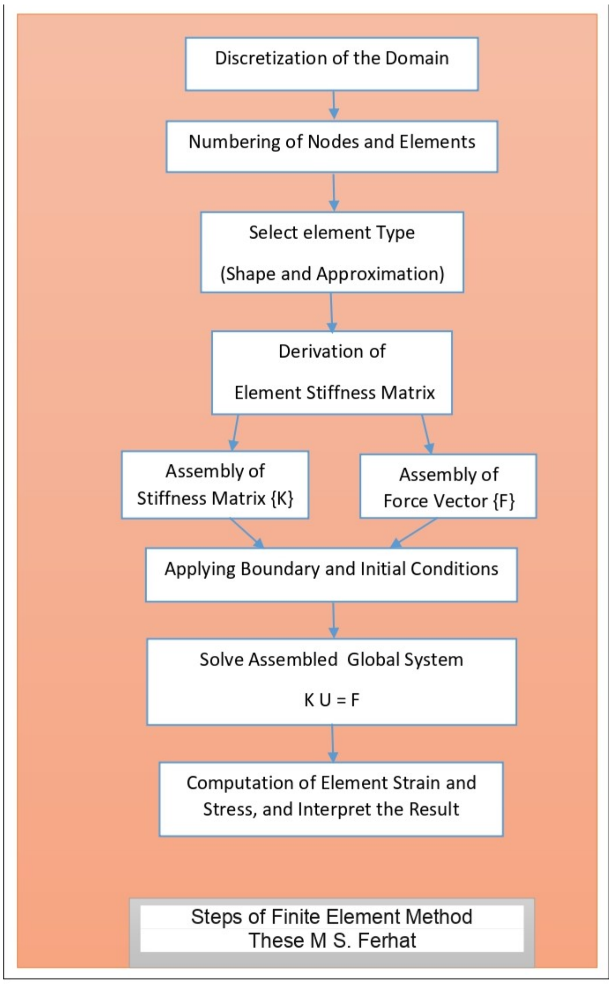


Figure 1.7: Steps of finite element method

We review only the material related to linear triangular elements, knowing that the discussion here can be extended to quadrilateral elements, higher-order elements, and to domains of dimensions other than two. We will assume for the purpose of simplicity that that $\overline{\Omega} \subset \mathbb{R}^2$ is a planar polygonal domain divided into a limited number of triangles $K \in \mathcal{T}^h$,, where $\bigcup_{K\in\mathcal{T}^h} K$ and for distinct $K_1, K_2 \in \mathcal{T}^h, K_1 \cap K_2$ is either empty, a common vertex, or a common side.

For an arbitrary element K, we denote

- $h_K = \text{diam}(K) = \max\{\|x-y\| : x, y \in K\}$
- $h = \max\{h_K, K \in \mathcal{T}^h\} =$ the discretization parameter
- ρ_K The largest sphere's diameter inscribed in K

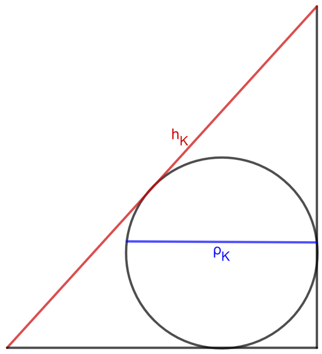


Figure 1.8: Dimensions in K

Definition 1.3.3 *A set of triangulations* $\left\{\mathcal{T}^h\right\}_h$ *of* $\overline{\Omega}$ *is said to be regular family if the discretization parameter* h *tends to zero while the size of the triangles decreases, and there exists a constant* σ*, independent of* h*, such that*

$$\forall K \in \mathcal{T}^h, \quad h_K \leq \sigma \rho_K.$$

We now consider approximations by linear finite element spaces. Denote by $\{x_i\}_{i=1}^{N_h} \subset \overline{\Omega}$ the set of the vertices of the elements in the partition $\mathcal{T}^h$, and let φ_i be the corresponding finite element basis function, which is linear on each element K, represented in Figure 1.9 and satisfies the Kronecker delta property

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad , 1 \leq i, j \leq N_h.$$

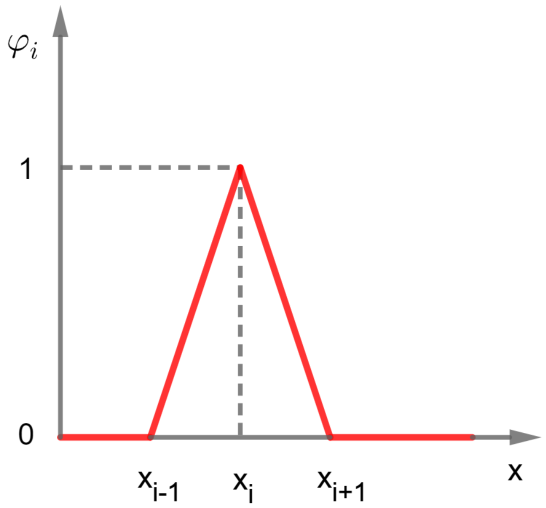


Figure 1.9: The Lagrange basis function

Let $\hat{K}_i$ be the patch of the elements K that contain x_i as a vertex, for each x_i. Then the basis function φ_i has a nonzero value only on $\hat{K}_i$. A continuous function $v \in C(\overline{\Omega})$ has the following finite element interpolant

$$\Pi^h v = \sum_{i=1}^{N_h} v(x_i)\varphi_i$$

We will need the following discrete versions of the Gronwall inequality to approximate the variational inequalities that arise in contact problems numerically.

Lemma 1.3.6 *Given $T > 0$, we define $k = T/N$ for a positive integer N. Assume that $\{f_n\}_{n=1}^N$ and $\{u_n\}_{n=1}^N$ are two sequences of nonnegative numbers that satisfy*

$$u_n \leq d_1 f_n + d_1 \sum_{j=1}^{n-1} k u_j, \quad n = 1, ..., N$$

for a positive constant d_1 that is independent of N and k. Then there exists a positive constant d_2, independent of N and k, such that

$$u_n \leq d_2 \left(f_n + \sum_{j=1}^{n-1} k f_j \right), \quad n = 1, ..., N.$$

Therefore, if k is sufficiently small. Then there exists a positive constant c, independent of N and k, such that

$$\max_{1\leq n\leq N} u_n \leq c \max_{1\leq n\leq N} f_n.$$

Chapter

2 Frictional Contact Problems with Normal Damped Response

Contents

2.1 Introduction

This chapter is composed of two sections

In the first section, we consider a general elastic-viscoplastic body in adhesive frictional contact with a reactive foundation, in the case when the external forces vary slowly and the quasistatic process is valid. An equation of first order ordinary differential type can be used to describe the evolution of the bonding field. The contact is described with a normal damped response and two internal variables that represent a temperature parameter and the damage of the contacting surface. In order to prove that the problem has an unique solution, we derive a variational formulation of the problem. The proof is provided by the use of evolution equations with monotone operators, a fixed point, and a standard existence and uniqueness result for parabolic inequalities.

In the second section, we suggest studying a frictional contact between two viscoelastic piezoelectric bodies with adhesion and damage. An inclusion of the parabolic type describes the progression of the damage. The process is quasistatic, the material's behavior is predicted by a nonlinear electro-viscoelastic constitutive law, and the contact is characterized by a normal damped response condition. The problem is formulated as a coupled system in displacement, stress, electric potential, adhesion, and damage field. We construct a variational formulation of the issue and demonstrate that the model has a single weak solution. The arguments used in the proofs come from evolution equations with monotone operators, a traditional existence and uniqueness conclusion based on parabolic inequalities, differential equations, and fixed point theory.

2.2 Contact Problem on Elasto-Viscoplasticity with Thermal Effects and Damage

Here, we look at a frictional contact problem between an elastic-viscoplastic body and a deformable foundation with a normal damped response and a local friction law in a quasistatic process. Two internal variables that can define a temperature parameter and the damage to the contacting surface are included in the model for the material. The problem is stated as a coupled system in terms of temperature, displacement, stress, and damage field. In order to prove that the model has a unique weak solution, we create a variational formulation of the problem. This section is divided into three paragraphs. In the first paragraph, we describe the mechanical issue before indicating the assumptions based on the information. The variational formulation of the mechanical problem is then provided in the second paragraph. Finally, in the third paragraph, we demonstrate that the model has a unique weak solution. The arguments of evolution equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities, and fixed point theory are used in the proofs.

2.2.1 Mechanical Problem and Assumptions

The physical setting corresponds to that introduced in Subsection 1.2.1 As a reminder, We take a domain Ω with a regular surface Γ that an elastic-thermo-viscoplastic body occupies. We next divide Γ into three disjoint measurable sections Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. Let $T > 0$ and $[0, T]$ represent the temporal interval under consideration. Assuming that the body is fixed on $\Gamma_1 \times (0, T)$, the displacement field is shown to vanish there. A volume force of density f_0 acts on $\Omega \times (0, T)$ and surface tractions of density f_2 operate on on $\Gamma_2 \times (0, T)$. We accept the possibility of an external heat source applied in$\Omega \times (0, T)$, according to the function q. We assume that the body forces and tractions vary gradually over time, thus we may ignore the system's accelerations, and the quasistatic approach to the process is adopted. Finally, the body comes in contact with a reactive foundation over the potential contact surface Γ_3. According to our assumption, the normal stress $\boldsymbol{\sigma}_\nu$ satisfies the condition for a general normal damped response.

$$-\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \tag{2.2.1}$$

where p_ν is a specified function and $\dot{\mathbf{u}}_\nu$ stands for the normal velocity, on which the normal pressure is often dependent, according to equality (2.2.1).

The resistance of the foundation to penetration is proportional to the normal velocity in the situation where $p_\nu(r) = kr$, with k ≥ 0. When simulating the motion of a deformable body on sand or another granular material, this type of behavior was taken into account in [49].

The frictional aspect of the contact leads to the selection of the following friction law:

$$-\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \tag{2.2.2}$$

where p_τ indicates the prescribed vector-valued function, $\dot{\mathbf{u}}_\tau$ represents the tangential velocity and $\boldsymbol{\sigma}_\tau$ is the tangential force acting on the contact boundary, A differential inclusion of the parabolic type is used to describe the evolution of the damage, and a thermo-elastic-viscoplastic constitutive law with damage is used to simulate the behavior of the material. The following is a typical formulation of the mechanical issue of the quasistatic contact with normal damped response under the above assumptions.

Problem P

Determine the stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$, the displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, the temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ and the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) + \mathcal{B}\big(\varepsilon(\mathbf{u}(t))\big) \\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big) ds, \quad \text{in } \Omega \times (0,T), \end{aligned} \tag{2.2.3}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{2.2.4}$$

$$\dot{\theta} - k_0 \Delta\theta = \psi\big(\boldsymbol{\sigma}, \varepsilon(\dot{\mathbf{u}}), \theta, \beta\big) + q, \quad \text{in } \Omega \times (0,T), \tag{2.2.5}$$

$$\dot{\beta} - k_1 \Delta\beta + \partial\varphi_K(\beta) \ni \phi\big(\boldsymbol{\sigma}, \varepsilon(\mathbf{u}), \theta, \beta\big), \quad \text{in } \Omega \times (0,T), \tag{2.2.6}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.2.7}$$

$$\boldsymbol{\sigma}\nu = f_2, \quad \text{on } \Gamma_2 \times (0,T), \tag{2.2.8}$$

$$-\boldsymbol{\sigma}_\nu = p_\nu(\dot{\mathbf{u}}_\nu), \quad -\boldsymbol{\sigma}_\tau = p_\tau(\dot{\mathbf{u}}_\tau), \quad \text{on } \Gamma_3 \times (0,T), \tag{2.2.9}$$

$$k_0 \frac{\partial\theta}{\partial\nu} + B\theta = 0, \quad \text{on } \Gamma \times (0,T), \tag{2.2.10}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \quad \text{on } \Gamma \times (0,T), \tag{2.2.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \beta(0) = \beta_0, \qquad \theta(0) = \theta_0, \quad \text{in } \Omega. \tag{2.2.12}$$

This problem illustrates the quasistatic progression of damage in thermoelastic viscoplastic materials. The thermo-elastic-viscoplastic constitutive law, which was already covered in the first section, is represented by Equation (2.2.3). Since we presume that the process is quasistatic, we employ the relation (2.2.4), which serves as the equilibrium equation. Energy conservation is represented by equation (2.2.5), where q is a specified volume heat source and ψ is a nonlinear constitutive function that reflects the heat produced by the work of internal forces. The evolution of the damage field is described in inclusion (2.2.6), which is determined by the source damage function ϕ, where $\partial\varphi_K$ represents the subdifferential of indicator function of the set K of acceptable damage functions.

The displacement and traction boundary conditions are specified in conditions (2.2.7) and (2.2.8), respectively. On Γ_3, (2.2.9) represents the normal damped response condition and its corresponding friction law. The temperature on Γ denotes a Fourier boundary condition in Equation Equation (2.2.10). An homogeneous Newman boundary condition for the damage field on Γ is represented by equation (2.2.11). The initial data are finally represented by the functions $\mathbf{u}_0$, β_0 and θ_0 in (2.2.12).

We require additional notation in order to achieve the variational formulation of the problem (2.2.3)–(2.2.12). Let V be the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 \,|\, \mathbf{v} = 0 \ \text{ on } \Gamma_1\}\,.$$

Due to the fact that $meas(\Gamma_1) \geq 0$ and Γ is Lipschitz, Korn's inequality states that there exists a positive constant C that solely depends on Ω and Γ_1 such that

$$|\varepsilon(\mathbf{v})|_{\mathcal{H}} \geq C\,|\mathbf{v}|_{H_1}\,, \quad \forall\,\mathbf{v} \in V. \tag{2.2.13}$$

We refer the reader to [47, p. 79] for the proof of this inequality

We consider the inner product of V and the corresponding norm provided by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall\,\mathbf{u}, \mathbf{v} \in V, \tag{2.2.14}$$

$$|\,\mathbf{v}\,|_V = |\varepsilon(\mathbf{v})|_{\mathcal{H}}\,, \quad \forall\,\mathbf{v} \in V. \tag{2.2.15}$$

Additionally, according to the Sobolev trace theorem and (2.2.13), we have a posetive constant $C_0 > 0$ that only depends on Ω, Γ_1, and Γ_3 so that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \leq C_0\,|\mathbf{v}|_V\,, \quad \forall \mathbf{v} \in V. \tag{2.2.16}$$

We take into account the following assumptions when studying the mechanical problem (2.2.3)–(2.2.12).

The plasticity operator $\mathcal{G} : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to S_d$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \qquad |\mathcal{G}(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \mathcal{G}(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \leq L_{\mathcal{G}}(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\ \qquad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\ \qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d,\ \forall\, \varepsilon_1, \varepsilon_2 \in S_d,\ \forall\, \theta_1, \theta_2 \in \mathbb{R},\ \forall\, \beta_1, \beta_2 \in \mathbb{R},\ a.e.\ x \in \Omega. \\ (b)\ \ x \mapsto \mathcal{G}(x, \sigma, \varepsilon, \theta, \beta) \text{ is a Lebesgue measurable mapping on } \Omega, \\ \qquad \forall\, \boldsymbol{\sigma}, \varepsilon \in S_d,\ \forall\, \theta, \beta \in \mathbb{R}. \\ (c)\ \ x \mapsto \mathcal{G}(x, 0, 0, 0, 0) \text{ is a map belonging to } \mathcal{H}. \end{cases} \tag{2.2.17}$$

The elasticity operator $\mathcal{B} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_{\mathcal{B}} > 0 \text{ such that} \\ \qquad |\mathcal{B}(x, \varepsilon_1) - \mathcal{B}(x, \varepsilon_2)| \leq L_{\mathcal{B}}(|\varepsilon_1 - \varepsilon_2|\,, \\ \qquad \forall\, \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega. \\ (b)\ \ x \mapsto \mathcal{B}(x, \varepsilon) \text{ is Lebesgue measurable mapping on } \Omega, \\ \qquad \forall\, \varepsilon \in S_d. \\ (c)\ \ x \mapsto \mathcal{B}(x, 0,) \text{ is a map belonging to } \mathcal{H}. \end{cases} \tag{2.2.18}$$

The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \qquad |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \le L_{\mathcal{A}} \, |\varepsilon_1 - \varepsilon_2| \, , \\ \qquad \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega. \\ (b) \text{ There is a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ \qquad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}} \, |\varepsilon_1 - \varepsilon_2|^2 \, , \\ \qquad \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \; a.e. \; x \in \Omega. \\ (c) \;\; x \mapsto \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable mapping on } \Omega, \\ \qquad \forall \, \varepsilon \in S_d. \\ (d) \text{ The mapping } x \mapsto \mathcal{A}(x, 0) \text{ is a map belonging to } \mathcal{H}. \end{cases} \tag{2.2.19}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \text{ Thereis a constant } L_\tau > 0 \text{ such that} \\ \qquad |p_\tau(x, r_1) - p_\tau(x, r_2)| \le L_\tau \, |r_1 - r_2| \quad \forall \, r_1, r_2 \in \mathbb{R}^d, \; a.e. \; x \in \Gamma_3. \\ (b) \; (p_\tau(x, r_1) - p_\tau(x, r_2)) \cdot (r_1 - r_2) \ge 0 \quad \forall \, r_1, r_2 \in \mathbb{R}^d, \; a.e. \; x \in \Gamma_3. \\ (c) \; x \mapsto p_\tau(x, r) \text{ is Lebesgue measurable mapping on } \Gamma_3, \\ \qquad \forall \, r \in \mathbb{R}^d. \\ (d) \;\; r \mapsto p_\tau(x, r) \text{ is a map continuous on } \mathbb{R}^d, \; a.e. \; x \in \Gamma_3. \\ (e) \; p_\tau(x, r).\nu(x) = 0 \;\; \forall r \in \mathbb{R}^d \text{ such that } r.\nu(x) = 0, \;\; a.e. \; x \in \Gamma_3. \end{cases} \tag{2.2.20}$$

The normal contact function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_\nu > 0 \text{ such that} \\ \qquad |p_\nu(x, r_1) - p_\nu(x, r_2)| \le L_\nu \, |r_1 - r_2| \,, \quad \forall \, r_1, r_2 \in \mathbb{R}, \, a.e. \; x \in \Gamma_3. \\ (b) \; (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \ge 0 \quad \forall \, r_1, r_2 \in \mathbb{R}, \, a.e. \; x \in \Gamma_3. \\ (c) \text{ The } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable mapping on } \Gamma_3, \\ \qquad \forall \, r \in \mathbb{R}. \\ (d) \;\; r \mapsto p_\nu(x, r) \text{ is a map continuous on } \mathbb{R}, \, a.e. \; x \in \Gamma_3. \end{cases} \tag{2.2.21}$$

The damage source function $\phi : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_\phi > 0 \text{ such that} \\ \qquad |\phi(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \phi(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \le L_\phi(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\ \qquad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\ \qquad \forall \, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d, \, \forall \, \varepsilon_1, \varepsilon_2 \in S_d, \, \forall \, \theta_1, \theta_2 \in \mathbb{R}, \, \forall \, \beta_1, \beta_2 \in \mathbb{R}, \, a.e. \; x \in \Omega. \\ (b) \;\; x \mapsto \phi(x, \boldsymbol{\sigma}, \varepsilon, \theta, \beta) \text{ is Lebesgue measurable mapping on } \Omega, \\ \qquad \forall \, \boldsymbol{\sigma}, \varepsilon \in S_d, \, \forall \, \theta, \beta \in \mathbb{R}. \\ (c) \;\; x \mapsto \phi(x, 0, 0, 0, 0) \text{ is a map belonging to } \mathcal{H}. \end{cases} \tag{2.2.22}$$

The nonlinear constitutive function $\psi : \Omega \times S_d \times S_d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There is a constant } L_\psi > 0 \text{ such that} \\ \qquad |\psi(x, \boldsymbol{\sigma}_1, \varepsilon_1, \theta_1, \beta_1) - \psi(x, \boldsymbol{\sigma}_2, \varepsilon_2, \theta_2, \beta_2)| \leq L_\psi(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| \\ \qquad + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\beta_1 - \beta_2|), \\ \qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in S_d,\ \forall\, \varepsilon_1, \varepsilon_2 \in S_d,\ \forall\, \theta_1, \theta_2 \in \mathbb{R},\ \forall\, \beta_1, \beta_2 \in \mathbb{R},\ a.e.\ x \in \Omega. \\ (b)\ \ x \mapsto \psi(x, \boldsymbol{\sigma}, \varepsilon, \theta, \beta) \text{ is Lebesgue measurable mapping on } \Omega, \\ \qquad \forall\, \boldsymbol{\sigma}, \varepsilon \in S_d,\ \forall\, \theta, \beta \in \mathbb{R}. \\ (c)\ \ x \mapsto \psi(x, 0, 0, 0, 0) \text{ is a map belonging to } \mathcal{H}. \end{cases} \tag{2.2.23}$$

We assume that the surface tractions and body forces satisfy

$$f_2 \in L^2(0, T; L^2(\Gamma_2)^d), \quad f_0 \in L^2(0, T; H). \tag{2.2.24}$$

The volume heat source q satisfies

$$q \in L^2(0, T; L^2(\Omega)). \tag{2.2.25}$$

We also assume that

$$k_i > 0 \quad (i = 0, 1), \qquad B > 0. \tag{2.2.26}$$

In end, we assume that the initial data meets the following conditions.

$$\mathbf{u}_0 \in V, \quad \theta_0 \in V, \quad \beta_0 \in K.. \tag{2.2.27}$$

Next, $f(t)$ represents the element of V' given by

$$(f(t), \mathbf{v})_{V' \times V} = \int_\Omega f_0(t).\mathbf{v}\, dx + \int_{\Gamma_2} f_2(t).\mathbf{v}\, da \quad \forall\, \mathbf{v} \in V. \tag{2.2.28}$$

We observe that the assumptions (2.2.24) imply

$$f \in L^2(0,T;V'). \tag{2.2.29}$$

The bilinear forms a_0 and a_1 are defined as follows

$$a_0 : V \times V \to \mathbb{R}, \quad a_0(\zeta, \xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi \, dx + B \int_\Gamma \zeta\xi d\gamma, \tag{2.2.30}$$

$$a_1 : V \times V \to \mathbb{R}, \quad a_1(\zeta, \xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi \, dx. \tag{2.2.31}$$

Let $j : V \times V \to \mathbb{R}$ be the functional defined by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(\mathbf{u}_\nu)\mathbf{v}_\nu \, da + \int_{\Gamma_3} p_\tau(\mathbf{u}_\tau) \cdot \mathbf{v}_\tau \, da, \quad \forall \mathbf{u}, \mathbf{v} \in V. \tag{2.2.32}$$

Let $t \in [0, T]$ and $\mathbf{v} \in V$. Using Green's formula (1.3.1) and (2.2.4) we obtain

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} = \int_\Omega f_0(t) \cdot \mathbf{v} \, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \mathbf{v} \, da, \quad \forall \mathbf{v} \in V. \tag{2.2.33}$$

As a result of applying the boundary conditions (2.2.7) and (2.2.8), we find

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \mathbf{v} \, da = \int_{\Gamma_2} f_2(t) \cdot \mathbf{v} \, da + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \mathbf{v} \, da. \tag{2.2.34}$$

From (2.2.33), (2.2.34) and (2.2.28) it is now obvious that

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} = (f(t), \mathbf{v})_{V' \times V} + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \mathbf{v} \, da, \quad \forall \mathbf{v} \in V. \tag{2.2.35}$$

On the other hand, from (1.2.1), (1.2.2) and (2.2.9) we have

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \mathbf{v} = -p_\nu(\dot{\mathbf{u}}_\nu(t))\mathbf{v}_\nu - p_\tau(\dot{\mathbf{u}}_\tau(t)) \cdot \mathbf{v}_\tau, \quad \text{on } \Gamma_3. \tag{2.2.36}$$

Finally from (2.2.32), (2.2.35) and (2.2.36) we find

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) = (f(t), \mathbf{v})_{V' \times V}.$$

2.2.2 Variational Formulation

The variational formulation of the quasistatic problem with normal damped response, friction, and damage can be constructed as follows.

Problem PV

Determine the stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, the displacement field $\mathbf{u} : [0,T] \to V$, the damage field $\beta : [0,T] \to H^1(\Omega)$ and the temperature $\theta : [0,T] \to V$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{B}(\varepsilon(\mathbf{u}(t)) + \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(t))\big) \\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}(s)), \theta(s), \beta(s)\Big) ds, \ \text{ a.e. } \ t \in (0,T)\,, \end{aligned} \tag{2.2.37}$$

$$(f(t), \mathbf{v})_{V' \times V} = (\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}), \quad \forall \mathbf{v} \in V, \ \forall t \in [0,T]\,, \tag{2.2.38}$$

$$\begin{aligned} &\big(\psi(\boldsymbol{\sigma}(t)), \varepsilon(\dot{\mathbf{u}}(t)), \theta(t), \beta(t), \mathbf{v}\big)_{V' \times V} + (q(t), \mathbf{v})_{V' \times V} = (\dot{\theta}(t), \mathbf{v})_{V' \times V} \\ &\quad + a_0(\theta(t), \mathbf{v}) \quad \forall \mathbf{v} \in V, \ \text{ a.e. } \ t \in (0,T)\,, \end{aligned} \tag{2.2.39}$$

$$\begin{aligned} &\beta(t) \in \boldsymbol{K}, \quad \big(\phi\big(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \beta(t), \xi - \beta(t)\big)_{L^2(\Omega)} \\ &\quad \le (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a_1(\beta(t), \xi - \beta(t)), \quad \forall \xi \in \boldsymbol{K}, \ \text{ a.e. } \ t \in (0,T)\,, \end{aligned} \tag{2.2.40}$$

$$\beta(0) = \beta_0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \qquad \theta(0) = \theta_0. \tag{2.2.41}$$

We note that the stress field, displacement field, temperature, and damage field are used to formulate the variational problem PV.

2.2.3 Existence and Uniqueness Results

The following is our main result, that we present here and justify in the paragraph below.

Theorem 2.2.1 *suppose (2.2.19)–(2.2.20) are true. Then the problem PV has a unique solution $\{\boldsymbol{\sigma}, \mathbf{u}, \beta, \theta\}$. Additionally, the solution has the regularity*

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{2.2.42}$$

$$\mathbf{u} \in C^1(0,T;V), \tag{2.2.43}$$

$$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{2.2.44}$$

$$\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;V), \tag{2.2.45}$$

The solution $\{\boldsymbol{\sigma}, \mathbf{u}, \beta, \theta\}$ that satisfies (2.2.37)–(2.2.41) is said to as a weak solution to the contact problem P. We come to conclusion that problem (2.2.3)–(2.2.12) has a unique weak solution satisfying (2.2.43)–(2.2.44), under the mentioned assumptions.

Proof of Theorem 2.1.1

Theorem 2.2.1 will be proved in steps using arguments from evolution equations with monotone operators, a standard existence and uniqueness result on parabolic inequalities, and fixed-point. For this reason, we'll assume in the next part that (2.2.19)–(2.2.20) are true and that, everywhere in this section, C will stand for a strictly positive integer whose value may vary depending on the problem's data but is independent of time. Additionally, we hide the explicit dependence of various functions on $\mathrm{x} \in \Omega \cup \Gamma$ in the following for the purpose of simplicity.

Let $\eta \in L^2(0,T;V')$ be provided . The following variational problem is considered in the first step.

Problem PV_η. Determine a displacement field $\mathbf{u}_\eta : [0,T] \to V$ so that

$$\big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_\eta(t))), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} + (\eta(t), \mathbf{v})_{V'\times V} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) = (f(t), \mathbf{v})_{V'\times V}, \tag{2.2.46}$$

$$\forall \mathbf{v} \in V \quad a.e.\ t \in (0,T),$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{2.2.47}$$

The study of Problem PV_η produced the next result.

Lemma 2.2.1 *Problem PV_η has an unique solution, and it satisfies the regularity given in (2.2.43). Additionally, if $\mathbf{u}_i$ is the solution to these problems PV_{η_i} for $\eta_i \in L^2(0,T;V')$, $i = 1, 2$ then there is a $C > 0$ such that*

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'} . \tag{2.2.48}$$

Proof. We use the Riesz's representation theorem to determine the element $f_\eta(t) \in V'$ and the operator $T : V \to V'$ by

$$(T\mathbf{u}, \mathbf{v})_{V' \times V} = (\mathcal{A}(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v}))_{\mathcal{H}} + j(\mathbf{u}, \mathbf{v}), \tag{2.2.49}$$

$$(f_\eta(t), \mathbf{v})_{V' \times V} = (f(t), \mathbf{v})_{V' \times V} - (\eta(t), \mathbf{v})_{V' \times V}, \tag{2.2.50}$$

for all $\mathbf{u}, \mathbf{v} \in V$, $t \in [0, T]$. Let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Using (2.2.49) and (2.2.32) we find

$$\begin{aligned}(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V' \times V} &= \big(\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} \big(p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu})\big)(\mathbf{u}_{1\nu} - \mathbf{u}_{2\nu}) da \\ &+ \int_{\Gamma_3} \big(p_\tau(\mathbf{u}_{1\tau}) - p_\tau(\mathbf{u}_{2\tau})\big).(\mathbf{u}_{1\tau} - \mathbf{u}_{2\tau}) da.\end{aligned}$$

Additionally, by taking into account (2.2.19)(b), (2.2.21)(b) and (2.2.20)(b) leads to

$$(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V' \times V} \geq m_{\mathcal{A}} |\mathbf{u}_1 - \mathbf{u}_2|_V^2 . \tag{2.2.51}$$

The relations (2.2.49) and (2.2.32) are used once more to find

$$\begin{aligned}(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{v})_{V' \times V} &= \big(\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} \\ &+ \int_{\Gamma_3} \big(p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu})\big)(\mathbf{v}_\nu) da + \int_{\Gamma_3} \big(p_\tau(\mathbf{u}_{1\tau}) - p_\tau(\mathbf{u}_{2\tau})\big).(\mathbf{v}_\tau) da.\end{aligned}$$

And, we use ((2.2.19)(a) to infer that

$$\forall\, \mathbf{u}_1, \mathbf{u}_2 \in V \quad |T\mathbf{u}_1 - T\mathbf{u}_2|_{V'} \leq L_{\mathcal{A}} \, |\mathbf{u}_1 - \mathbf{u}_2|_V \,. \tag{2.2.52}$$

The operator $T : V \to V'$ is a strongly monotone, as established by inequality (2.2.51). Inequality(2.2.52) also implies that T is a Lipschitz continuous. As a result, using a standard result for nonlinear equations (see, for example, [12]), there exists a singular element $\mathbf{w}_\eta$ which satisfies

$$T\mathbf{w}_\eta(t) = f_\eta(t) \quad a.e.\; t \in (0, t), \tag{2.2.53}$$

$$\mathbf{w}_\eta \in C(0, T; V), \tag{2.2.54}$$

Now we define the function $\mathbf{u}_\eta : [0, T] \to V$ by

$$\mathbf{u}_\eta = \int_0^t \mathbf{w}_\eta(s) ds + \mathbf{u}_0. \tag{2.2.55}$$

The relations (2.2.49), (2.2.53)–(2.2.55) imply that $\mathbf{u}_\eta$ is a solution of the equation (2.2.46) and it satisfies (2.2.43).

Now let's look at the proof of estimate (2.2.48). For this purpose we use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$ for $i = 1, 2.$ and let $\eta_1, \eta_2 \in L^2(0, T, V')$

We use (2.2.46) and subtract the two obtained equations, we choose $\mathbf{v} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$ as test function to find

$$\begin{aligned} &\Big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}_1(t))) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}_2(t))), \varepsilon(\dot{\mathbf{u}}_1(t)) - \varepsilon(\dot{\mathbf{u}}_2(t))\Big)_{\mathcal{H}} \\ &+ j\big(\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big) - j\big(\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big) \\ &= \big(\eta_2(t) - \eta_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big)_{V' \times V}, \quad \forall t \in [0, T] \,. \end{aligned} \tag{2.2.56}$$

Considering (2.2.15) and (2.2.19)(b) we conclude that

$$\Big(\mathcal{A}(\varepsilon(\mathbf{w}_1(t))) - \mathcal{A}(\varepsilon(\mathbf{w}_2(t))), \varepsilon(\mathbf{w}_1(t)) - \varepsilon(\mathbf{w}_2(t))\Big)_{\mathcal{H}} \qquad (2.2.57)$$
$$\geq C\,|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V^2\,, \quad \forall t \in [0,T]\,.$$

From (2.2.32), (2.2.21) and (2.2.20), we find

$$j(\mathbf{w}_1(t), \mathbf{w}_1(t) - \mathbf{w}_2(t)) - j(\mathbf{w}_2(t), \mathbf{w}_1(t) - \mathbf{w}_2(t)) \geq 0, \quad \forall t \in [0,T]. \qquad (2.2.58)$$

The Cauchy-Schwartz inequality also allows us to obtain

$$\big(\eta_2(t) - \eta_1(t), \dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\big)_{V' \times V} \leq |\eta_2(t) - \eta_1(t)|_{V'}\,|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V\,. \qquad (2.2.59)$$

We Combine (2.2.56)–(2.2.59) with some algebraic manipulations to obtain

$$|\mathbf{w}_1(t) - \mathbf{w}_2(t)|_V \leq C\,|\eta_1(t) - \eta_2(t)|_{V'}\,. \qquad (2.2.60)$$

Given that $\mathbf{u}_1(0) = \mathbf{u}_2(0)$ and $\mathbf{u}_i(t) = \int_0^t \mathbf{w}_i(s)ds + \mathbf{u}_0$ we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \leq \int_0^t |\mathbf{w}_1(s) - \mathbf{w}_2(s)|_V\,ds.$$

We deduce from the two previous inequalities the estimate (2.2.48). ■

Let $\chi \in L^2(0,T,V')$ be provided. The following intermediate variational problem is considered in the second step.

Problem PV_χ

Determine the temperature $\theta_\chi : [0,T] \to V$ such that

$$\big(\chi(t) + q(t), \mathbf{v}\big)_{V' \times V} = (\dot{\theta}_\chi(t), \mathbf{v})_{V' \times V} + a_0(\theta_\chi(t), \mathbf{v}) \tag{2.2.61}$$

$$\forall \mathbf{v} \in V, \quad a.e.\ t \in (0,T)\,,$$

$$\theta_\chi(0) = \theta_0. \tag{2.2.62}$$

Lemma 2.2.2 *The auxiliary Problem PV_η has an unique solution θ_χ , and it satisfies the regularity given in (2.2.45). Additionally, if θ_i is the solution to these problems PV_{χ_i} for $\chi_i \in L^2(0,T;V')$, $i = 1,2$ then there is $C > 0$ such that*

$$|\theta_1(t) - \theta_2(t)|_V^2 \le C \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2\, ds. \tag{2.2.63}$$

Proof. We Use the definition of the bilinear form a_0 given in (2.2.30) to obtain

$$a_0(\xi,\xi) = k_0 \int_\Omega |\nabla \xi|^2\ dx + B \int_\Gamma |\xi|^2\, d\gamma.$$

By applying the Friedrichs-Poincaré inequality, there is a constant $F > 0$ such that

$$\int_\Omega |\nabla \xi|^2\ dx + \frac{B}{k_0} B \int_\Gamma |\xi|^2\, d\gamma \ge F \int_\Omega |\xi|^2\ dx.$$

Consequently, a constant $C > 0$ exists that satisfies

$$a_0(\xi,\xi) \ge C\, |\xi|_{V'}^2\,, \quad \forall \xi \in V,$$

It shows that a_0 is V-elliptic. Thus, the variational equation (2.2.61) has a unique solution θ_χ that is satisfying (2.2.45) based on traditional considerations of functional analysis concerning parabolic equations .

Now let's look at the proof of estimate (2.2.63). For this purpose we use the notation $\theta_\chi = \theta_i$ for $i = 1, 2.$ and let $\chi_1, \chi_2 \in L^2(0,T;V')$. We use (2.2.61) and subtract the two obtained equations, we choose $\mathbf{v} = \theta_1 - \theta_2$ as test function to find

$$(\dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2)_{V'\times V} + a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) = (\chi_1 - \chi_2, \theta_1 - \theta_2)_{V'\times V}, \quad a.e.\ t \in (0,T). \tag{2.2.64}$$

Considering the inequality $a_0(\theta_1 - \theta_2, \theta_1 - \theta_2) \geq 0$ we discover that

$$(\dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2)_{V'\times V} \leq (\chi_1 - \chi_2, \theta_1 - \theta_2)_{V'\times V}. \tag{2.2.65}$$

We find using Cauchy–Schwartz inequality

$$(\dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2)_{V'\times V} \leq |\chi_1 - \chi_2|_{V'} |\theta_1 - \theta_2|_V. \tag{2.2.66}$$

Using initial conditions $\theta_1(0) = \theta_2(0) = \theta_0$, we integrate the previous inequality with regard to time and arrive to the conclusion that

$$\frac{1}{2} |\theta_1(t) - \theta_2(t)|_V^2 \leq \int_0^t |\chi_1(s) - \chi_2(s)|_{V'} |\theta_1(s) - \theta_2(s)|_V \, ds. \tag{2.2.67}$$

Applying the inequality: $2ab \leq a^2 + b^2 \quad \forall a, b \in \mathbb{R}$ after multiplying the members of the previous inequality by 2, allow us to find

$$|\theta_1(t) - \theta_2(t)|_V^2 \leq \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2 \, ds + \int_0^t |\theta_1(s) - \theta_2(s)|_V^2 \, ds.$$

Now from a Gronwall-type argument, it follows that

$$|\theta_1(t) - \theta_2(t)|_V^2 \leq C \int_0^t |\chi_1(s) - \chi_2(s)|_{V'}^2 \, ds. \tag{2.2.68}$$

■

Let $\mu \in L^2(0,T;L^2(\Omega)$ be provided. The following variational problem for the damage field is considered in the third step.

Problem PV_μ

Determine the damage field $\beta_\mu : [0,T] \to H^1(\Omega)$ such that

$$\beta_\mu(t) \in K, \quad \big(\dot{\beta}_\mu(t), \xi - \beta_\mu(t)\big)_{L^2(\Omega)} + a_1\big(\beta_\mu(t), \xi - \beta_\mu(t)\big) \tag{2.2.69}$$
$$\geq \big(\mu(t), \xi - \beta(t)\big)_{L^2(\Omega)} \quad \forall \xi \in V,\ a.e.\ t \in (0,T)\,,$$

$$\beta_\mu(0) = \beta_0. \tag{2.2.70}$$

We apply Theorem 1.3.7 to problem PV_μ.

Lemma 2.2.3 *The auxiliary Problem PV_μ has an unique solution β_μ , and it satisfies the regularity given in (2.2.44). Additionally, if β_i is the solution to these problems PV_{μ_i} for $\mu_i \in L^2(0,T;L^2(\Omega))$, $i = 1,2$ then there is $C > 0$ such that*

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq C \int_0^t |\mu_1(s) - \mu_2(s)|^2_{L^2(\Omega)}\, ds. \tag{2.2.71}$$

Proof. $L^2(\Omega)$ is associated with $(L^2(\Omega))'$ and it is identified with a subspace of $(H^1(\Omega))'$, where $(L^2(\Omega))'$ and $(H^1(\Omega))'$ represent the dual of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. The inclusion map is continuous and $H^1(\Omega)$ is dense in $L^2(\Omega)$. The duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ is denoted by the notation $(.,.)_{(H^1(\Omega))'\times H^1(\Omega)}$, we can write

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'$$
$$(\beta,\xi)_{(H^1(\Omega))'\times H^1(\Omega)} = (\beta,\xi)_{L^2(\Omega)} \quad \forall \xi \in H^1(\Omega).$$

Furthermore, we observe that K is a closed convex set in $H^1(\Omega)$. The definition of the bilinear form a_1 in (2.2.31), along with the knowledge that $\beta_\mu \in K$ in (2.2.27), make it simple to see that lemma 2.2.3 is a direct consequence of Theorem 1.3.7.

Now let's look at the proof of estimate (2.2.71). For this purpose we use the notation $\beta_\mu = \beta_i$ for $i = 1, 2$, and let $\mu_1, \mu_2 \in L^2(0,T; L^2(\Omega))$, we use (2.2.69) and subtract the two obtained equations, we choose $\xi = \beta_1 - \beta_2$ as test function to find

$$(\mu_1 - \mu_2, \beta_1 - \beta_2)_{L^2(\Omega)} \geq (\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \quad a.e.\ t \in (0,T).$$

Considering that $a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \geq 0$, we arrive at

$$(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} \leq (\mu_1 - \mu_2, \beta_1 - \beta_2)_{L^2(\Omega)}. \tag{2.2.72}$$

Using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and integrating the previous inequality with respect to time and , Permit us to obtain

$$\frac{1}{2} |\beta_1(t) - \beta_2(t)|^2 L^2(\Omega) \leq \int_0^t (\mu_1 - \mu_2, \beta_1 - \beta_2)_{L^2(\Omega)} ds. \tag{2.2.73}$$

We use inequalities of Hölder and Young to find

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq \int_0^t |\mu_1(s) - \mu_2(s)|^2_{L^2(\Omega)}\, ds + \int_0^t |\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)}\, ds.$$

Combining the previous inequality with Gronwall's inequality lead to

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} \leq C \int_0^t |\mu_1(s) - \mu_2(s)|^2_{L^2(\Omega)}\, ds. \tag{2.2.74}$$

■

In the last step, we create the following problem for the stress field using $\mathbf{u}_\eta, \theta_\chi$ and β_μ the values obtained above.

Problem $PV_{\eta\chi\mu}$**.** Determine the stress field $\boldsymbol{\sigma}_{\eta\chi\mu} : [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_{\eta\chi\mu}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big) ds,$$
$$\forall t \in [0,T]\,. \tag{2.2.75}$$

To study Problem $PV_{\eta\chi\mu}$ we have this result.

Lemma 2.2.4 *The Problem $PV_{\eta\chi\mu}$ has an unique solution , and it satisfies $\boldsymbol{\sigma}_{\eta\chi\mu} \in W^{1,2}(0,T;\mathcal{H})$. Additionally, if $\mathbf{u}_i$, θ_i, β_i and $\boldsymbol{\sigma}_i$, is the solutions of these problems PV_{η_i}, PV_{χ_i}, PV_{μ_i} and $PV_{\eta_i\chi_i\mu_i}$, respectively, for $(\eta_i, \chi_i, \mu_i) \in L^2(0,T;V' \times V' \times L^2(\Omega))$, $i = 1,2$, then there is $C > 0$ such that $\forall t \in [0,T]$*

$$|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \le C\Bigg(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V \tag{2.2.76}$$
$$+ \int_0^t (|\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V + |\theta_1(s) - \theta_2(s)|^2_V + |\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)}) ds\Bigg).$$

Proof. We consider the operator $\Lambda_{\eta\chi\mu} : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$

$$\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big) ds,$$
$$\forall \boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}),\ \forall t \in (0,T)\,. \tag{2.2.77}$$

We us the inequality of Holder, hypothesis (2.2.17) and (2.2.77) to obtain for $\boldsymbol{\sigma}_i \in L^2(0,T;\mathcal{H}),\ i = 1,2,$

$$|\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\, ds, \quad \forall t \in (0,T)\,.$$

It follows that

$$\begin{aligned} |\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_1(t) - \Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} &\le \left(L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\, ds \right)^2 \\ &\le (L_{\mathcal{G}})^2 T \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}\, ds \\ &\le C \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}\, ds. \end{aligned}$$

We can conclude from lemma 1.3.1 that exists a unique element $\boldsymbol{\sigma}_{\eta\chi\mu} \in L^2(0,T;\mathcal{H})$ which satisfies $\Lambda_{\eta\chi\mu}\boldsymbol{\sigma}_{\eta\chi\mu} = \boldsymbol{\sigma}_{\eta\chi\mu}$, therefore $\boldsymbol{\sigma}_{\eta\chi\mu}$ is the unique solution of problem $PV_{\eta\chi\mu}$.

we use the regularity of $\mathbf{u}_\eta, \theta_\chi, \beta_\mu$, and (2.2.77), (2.2.18), (2.2.17) to see that $\boldsymbol{\sigma}_{\eta\chi\mu} \in W^{1,2}(0.T;\mathcal{H})$. Consider now $(\eta_1,\chi_1,\mu_1),(\eta_2,\chi_2,\mu_2) \in L^2(0,T;V'\times V'\times L^2(\Omega))$, and for $i=1,2$ denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\theta_{\chi_i} = \theta_i$, $\beta_{\mu_i} = \beta_i$ and $\boldsymbol{\sigma}_{\eta_i\chi_i\mu_i} = \sigma_i$. Thus we have

$$\boldsymbol{\sigma}_i = \mathcal{B}(\varepsilon(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_i(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}_i(s))\big), \varepsilon(\mathbf{u}_i(s)), \theta_i(s), \beta_i(s)\Big)ds, \quad \forall t \in (0,T)\,,$$

and using the properties (2.2.18) and (2.2.17) of the operators $\mathcal{B}$ and $\mathcal{G}$, we find

$$\begin{aligned} |\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \le C\Bigg(& |\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V \\ & + \int_0^t (|\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V + |\theta_1(s) - \theta_2(s)|^2_V)ds \\ & + \int_0^t (|\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)} + |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}})ds\Bigg), \quad \forall t \in [0,T]\,. \end{aligned} \tag{2.2.78}$$

We use now a Gronwall argument in the previous inequality and deduce (2.2.76), which concludes the proof. ■

In view of these results and utilizing the characteristics of the operators $\mathcal{B}$ and $\mathcal{G}$, as well as the functions ψ and ϕ, we may finally consider the operator

$$\begin{aligned} &\mathcal{L} : L^2\big(0,T;V'\times V'\times L^2(\Omega)\big) \to L^2\big(0,T;V'\times V'\times L^2(\Omega)\big) \\ &\mathcal{L}(\eta,\chi(,\mu)(t) = \big(\mathcal{L}_1(\eta,\chi,\mu)(t), \mathcal{L}_2(\eta,\chi,\mu)(t), \mathcal{L}_3(\eta,\chi,\mu)(t)\big), \end{aligned} \tag{2.2.79}$$

where, for all $t \in [0,T]$ the mappings $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are given by

$$\begin{aligned} \big(\mathcal{L}_1(\eta,\chi,\mu)(t), \mathbf{v}\big)_{V'\times V} = \Bigg(& \mathcal{B}\big(\varepsilon(\mathbf{u}_\eta(t))\big) \\ & + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big)ds, \varepsilon(\mathbf{v})\Bigg)_{\mathcal{H}}, \end{aligned} \tag{2.2.80}$$

$$\mathcal{L}_2(\eta,\chi,\mu)(t) = \psi\big(\boldsymbol{\sigma}_{\eta\chi\mu}(t), \varepsilon(\dot{\mathbf{u}}_\eta(t)), \theta_\chi(t), \beta_\mu(t)\big). \tag{2.2.81}$$

$$\mathcal{L}_3(\eta,\chi,\mu)(t) = \phi\big(\boldsymbol{\sigma}_{\eta\chi\mu}(t), \varepsilon(\mathbf{u}_\eta(t)), \theta_\chi(t), \beta_\mu(t)\big). \tag{2.2.82}$$

Let $(\eta, \chi, \mu) \in L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$, we have the following result.

Lemma 2.2.5 *The operator $\mathcal{L}$ has a unique fixed point, in other words, there exists a unique element $(\eta^*, \chi^*, \mu^*) \in L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$ such that $\mathcal{L}(\eta^*, \chi^*, \mu^*) = (\eta^*, \chi^*, \mu^*)$.*

Proof. It is evident from Lemmas 2.2.1, 2.2.2, 2.2.3 and 2.2.4, that the operator $\mathcal{L}$ has a clear definition and takes values in $L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$.

Let $(\eta_1, \chi_1, \mu_1), (\eta_2, \chi_2, \mu_2) \in L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$ and $t \in [0, T]$, we use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\theta_{\chi_i} = \theta_i$, $\beta_{\mu_i} = \beta_i$ and $\boldsymbol{\sigma}_{\eta_i\chi_i\mu_i} = \boldsymbol{\sigma}_i$, for $i = 1, 2$.

We (2.2.16), (2.2.18), (2.2.17) and elementary algebraic manipulations to discover that

$$\begin{aligned}\big|\mathcal{L}_1(\eta_1, \chi_1, \mu_1)(t) - \mathcal{L}_1(\eta_2, \chi_2, \mu_2)(t)\big|_{V'}^2 \leq C\Bigg(&|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \\ &+ \int_0^t \big(|\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2 + |\theta_1(s) - \theta_2(s)|_V^2 \big) ds \\ &+ \int_0^t (|\beta_1(s) - \beta_2(s)|_{L^2(\Omega)}^2 + |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2) ds\Bigg).\end{aligned} \tag{2.2.83}$$

Additionally, using assumptions (2.2.23) on ψ, the definition (2.2.81), we find

$$\begin{aligned}&\big|\mathcal{L}_2(\eta_1, \chi_1, \mu_1)(t) - \mathcal{L}_2(\eta_2, \chi_2, \mu_2)(t)\big|_{V'}^2 \\ &\leq \big|\psi\big(\boldsymbol{\sigma}_1(t), \varepsilon(\dot{\mathbf{u}}_1(t)), \theta_1(t), \beta_1(t)\big) - \psi\big(\boldsymbol{\sigma}_2(t), \varepsilon(\dot{\mathbf{u}}_2(t)), \theta_2(t), \beta_2(t)\big)\big|_{V'}^2 \\ &\leq C\Bigg(|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2 + |\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)|_V^2 + |\theta_1(t) - \theta_2(t)|_V^2 \\ &\qquad + |\beta_1(t) - \beta_2(t)|_{L^2(\Omega)}^2 \Bigg).\end{aligned} \tag{2.2.84}$$

Similarly, using definition (2.2.82), assumptions (2.2.22) on ϕ we find

$$\begin{aligned}&\big|\mathcal{L}_3(\eta_1, \chi_1, \mu_1)(t) - \mathcal{L}_3(\eta_2, \chi_2, \mu_2)(t)\big|_{L^2(\Omega)}^2 \\ &\leq \big|\phi\big(\boldsymbol{\sigma}_1(t), \varepsilon(\mathbf{u}_1(t)), \theta_1(t), \beta_1(t)\big) - \phi\big(\boldsymbol{\sigma}_2(t), \varepsilon(\mathbf{u}_2(t)), \theta_2(t), \beta_2(t)\big)\big|_{L^2(\Omega)}^2, \\ &\leq C\Bigg(|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2 + |\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \\ &\qquad + |\theta_1(t) - \theta_2(t)|_V^2 + |\beta_1(t) - \beta_2(t)|_{L^2(\Omega)}^2 \Bigg).\end{aligned} \tag{2.2.85}$$

It follows from (2.2.83)–(2.2.85) and (2.2.79) that

$$\begin{aligned}\Big|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)} &\le C\bigg(|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}}\\ &+|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)|^2_V+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V\\ &+|\theta_1(t)-\theta_2(t)|^2_V+|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\\ &+\int_0^t(|\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V+|\theta_1(s)-\theta_2(s)|^2_V)ds\\ &+\int_0^t(|\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)}+|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}})ds\bigg).\end{aligned} \tag{2.2.86}$$

Inserting (2.2.76) in (2.2.86) yields

$$\begin{aligned}&\Big|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}\\ &\le C\bigg(|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)|^2_V+|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V+|\theta_1(t)-\theta_2(t)|^2_V+|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)}\\ &+\int_0^t(|\mathbf{u}_1(s)-\mathbf{u}_2(s)|^2_V+|\theta_1(s)-\theta_2(s)|^2_V+|\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)})ds\bigg).\end{aligned} \tag{2.2.87}$$

However, by applying (2.2.48) we may deduce that there exists $C>0$ such that

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V\le C\int_0^t|\eta_1(s)-\eta_2(s)|^2_{V'}\,ds. \tag{2.2.88}$$

Taking into account that $|\mathbf{w}_1(t)-\mathbf{w}_2(t)|_V \le C\,|\eta_1(t)-\eta_2(t)|_{V'}$, applying the estimates (2.2.63), (2.2.71), (2.2.88) and substituting in (2.2.87) we find

$$\begin{aligned}&\Big|\mathcal{L}(\eta_1,\chi_1,\mu_1)(t)-\mathcal{L}(\eta_2,\chi_2,\mu_2)(t)\Big|^2_{L^2\left(0,T;V'\times V'\times L^2(\Omega)\right)}\\ &\le C\bigg(\int_0^t\big(|\eta_1(s)-\eta_2(s)|^2_{V'}+|\chi_1(s)-\chi_2(s)|^2_{V'}+|\mu_1(s)-\mu_2(s)|^2_{L^2(\Omega)}\big)ds\\ &+\int_0^t\int_0^s\big(|\eta_1(s)-\eta_2(s)|^2_{V'}+|\chi_1(s)-\chi_2(s)|^2_{V'}+|\mu_1(s)-\mu_2(s)|^2_{L^2(\Omega)}\big)dr\,ds\bigg).\end{aligned} \tag{2.2.89}$$

Using some algebraic manipulations, leads us to conclude that there exists $C > 0$ such that

$$\begin{aligned}\Big|\mathcal{L}(\eta_1, \chi_1, \mu_1)(t) - \mathcal{L}(\eta_2, \chi_2, \mu_2)(t)\Big|^2_{L^2\big(0,T;V'\times V'\times L^2(\Omega)\big)} \\ \le C\int_0^t \Big|(\eta_1, \chi_1, \mu_1)(t) - (\eta_2, \chi_2, \mu_2)(t)\Big|^2_{L^2\big(0,T;V'\times V'\times L^2(\Omega)\big)} ds. \qquad (2.2.90)\end{aligned}$$

We conclude that the operator $\mathcal{L}$ has a unique fixed point in $L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$ by applying Lemma 1.3.1. This means that there exists a unique element $(\eta^*, \chi^*, \mu^*) \in L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$ such that $\mathcal{L}(\eta^*, \chi^*, \mu^*) = (\eta^*, \chi^*, \mu^*)$. ■

We now have all the ingredients to prove the Theorem 2.2.1 which we now complete .

Proof of Theorem 2.2.1. Let $\mathbf{u}_{\eta^*}$, θ_{χ^*}, β_{μ^*}, $\boldsymbol{\sigma}_{\eta^*\chi^*\mu^*}$ be the solutions of the problems PV$_\eta$, PV$_\chi$, PV$_\mu$ and PV$_{\eta\chi\mu}$ respectively, for $\mathcal{L}(\eta, \chi, \mu) = (\eta^*, \chi^*, \mu^*) \in L^2\big(0, T; V' \times V' \times L^2(\Omega)\big)$ and we denote

$$\mathbf{u} = \mathbf{u}_{\eta^*}, \quad \boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \boldsymbol{\sigma}_{\eta^*\chi^*\mu^*}, \qquad (2.2.91)$$

$$\theta = \theta_{\chi^*}, \quad \beta = \beta_{\mu^*}. \qquad (2.2.92)$$

We will demonstrate that the quadruple $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \beta\}$ is the unique solution of Problem PV. Indeed, we we utilize (2.2.91) to prove that (2.2.37) is satisfied, and we write (2.2.75) for $\eta = \eta^*, \chi = \chi^*$, and $\mu = \mu^*$. Then we use the first equality in (2.2.91) and (2.2.46) for $\eta = \eta^*$ to obtain

$$\big(\mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))), \varepsilon(\mathbf{v})\big)_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) + (\eta^*(t), \mathbf{v})_{V'\times V} = (f(t), \mathbf{v})_{V'\times V}, \qquad (2.2.93)$$
$$\forall \mathbf{v} \in V \quad a.e.\ t \in (0, T).$$

Combining (2.2.80) with $\mathcal{L}_1(\eta^*, \chi^*, \mu^*) = \eta^*$ results in

$$\begin{aligned}(\eta^*(t), \mathbf{v})_V = \mathcal{B}\big(\varepsilon(\mathbf{u}_\eta(t))\big)_{\mathcal{H}} \\ + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta\chi\mu}(s) - \mathcal{A}\big(\varepsilon(\dot{\mathbf{u}}_\eta(s))\big), \varepsilon(\mathbf{u}_\eta(s)), \theta_\chi(s), \beta_\mu(s)\Big) ds. \qquad (2.2.94)\end{aligned}$$

We replace (2.2.94) in (2.2.93) and utilize (2.2.37) to arrive at the conclusion that (2.2.38) is satisfied.

In addition, using equalities$\mathcal{L}_2(\eta^*, \chi^*, \mu^*) = \chi^*$ and $\mathcal{L}_3(\eta^*, \chi^*, \mu^*) = \mu^*$ in combination with (2.2.81) and (2.2.82), we discover that

$$\chi^* = \psi\big(\boldsymbol{\sigma}(t), \varepsilon(\dot{\mathbf{u}}(t)), \theta(t), \beta(t)\big). \tag{2.2.95}$$

$$\mu^* = \phi\big(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{u}(t)), \theta(t), \beta(t)\big). \tag{2.2.96}$$

Using the first equality in (2.2.92) and writing (2.2.61) for $\chi = \chi^*$ leads to

$$(\dot{\theta}(t), \mathbf{v})_{V'\times V} + a_0(\theta(t), \mathbf{v}) = (\chi^*(t) + q(t), \mathbf{v})_{V'\times V}, \tag{2.2.97}$$
$$\forall \mathbf{v} \in V \quad a.e.\ t \in (0, T)\,.$$

We replace (2.2.95) in (2.2.97) to arrive at the conclusion that (2.2.39) is satisfied.

Using the second equality in (2.2.92) and writing (2.2.69) for $\mu = \mu^*$ leads to

$$\big(\dot{\beta}(t), \xi - \beta(t)\big)_{L^2(\Omega)} + a_1\big(\beta(t), \xi - \beta(t)\big) \geq \big(\mu^*(t), \xi - \beta(t)\big)_{L^2(\Omega)},$$
$$\forall \xi \in V,\ a.e.\ t \in (0, T)\,. \tag{2.2.98}$$

Now that (2.2.98) and (2.2.96) have been combined, we can verify that (2.2.40) is satisfied, which completes the existence part of Theorem 2.2.1. The uniqueness part of Theorem 2.2.1 is a consequence of the unique solution of the problems PV_η, PV_χ, PV_μ and $\mathrm{PV}_{\eta\chi\mu}$ and the uniqueness of the fixed point of the operator $\mathcal{L}$, which concludes the proof. ■

2.3 Contact Problem on Vicoelasto-Piezoelectriicity with Adhesion and Damage

The frictional contact between two viscoelastic piezoelectric bodies with adhesion and damage is described by a mathematical model that we take into consideration. A parabolic type inclusion illustrates the evolution of the damage. The process is quasistatic, the material's behavior is modeled with a nonlinear electro-viscoelastic constitutive law, and the contact is described with a normal damped response condition.

In order to prove that the model has a unique weak solution, we create a variational formulation of the problem. The reasons used in the proofs are derived from evolution equations with monotone operators, a classical existence and uniqueness result based on parabolic inequalities, differential equations, and fixed point. This section is divided into three paragraphs. In the first paragraph, we start by the mechanical problem and assumptions. Then, in the second paragraph we give the variational formulation of the mechanical problem. Finally, in the third paragraph, We demonstrate that the model has a unique weak solution.

2.3.1 Mechanical Problem and Assumptions

The physical setting corresponds to that introduced in Subsection 1.2.1 As a reminder, We take into account two electro viscoelastic bodies that are located in two bounded domainsΩ^1, Ω^2, of the space $\mathbb{R}^d$ $(d = 2, 3)$. We use the superscript l to show that the quantity is associated with the Ω^l. The superscript l in the following varies between 1 and 2. The boundary Γ^l of each domain Ω^l is assumed to be Lipschitz continuous and is divided into three disjoint measurable parts, l 1, l 2, and l 3, on the one hand, parts Γ_1^l, Γ_2^l and Γ_3^l on one hand, and on two measurable parts Γ_a^l and Γ_b^l, on the other hand, such that $meas\left(\Gamma_1^l\right) > 0$, $meas\left(\Gamma_a^l\right) > 0$. The Ω^l body is submitted to f_0^l forces and volume electric charges of density q_0^l . The bodies are assumed to be clamped on Γ_1^l, so the displacement field vanishes there. The surface tractions f_2^l act on Γ_2^l. We also assume that the electrical potential vanishes on Γ_a^l and a surface electric charge of density q_2^l is prescribed on Γ_b^l. The two bodies are in contact along the common part $\Gamma_3^1 = \Gamma_3^2$, which will be denoted Γ_3 below. The bodies is in adhesive contact, over the contact surface Γ_3, We assume that the normal stress $\boldsymbol{\sigma}_\nu$ satisfies a general normal damped response condition with adhesion

$$-\boldsymbol{\sigma}_\nu = p_\nu([\dot{\mathbf{u}}_\nu]) - \gamma_\nu \beta^2 R_\nu([\mathbf{u}_\nu]),$$

where γ_ν represents the adhesion coefficient, assumed positive, $\dot{\mathbf{u}}_\nu$ represents the normal velocity and $[\mathbf{u}_\nu] = \mathbf{u}_\nu^1 + \mathbf{u}_\nu^2$. p_ν is a prescribed function, as an example, we may consider $p_\nu(r) = kr$, with $k \geq 0$, the resistance of the foundation to penetration is proportional to the normal velocity. This type of behavior was considered in [**?**] when modeling the motion of a deformable body on sand or a granular material. The associated friction law is chosen as follows

$$-\boldsymbol{\sigma}_\tau = p_\tau([\dot{\mathbf{u}}_\tau]) + q_\tau(\beta) R_\tau([\mathbf{u}_\tau]),$$

where $\boldsymbol{\sigma}_\tau$ represents the tangential force on the contact boundary, $\dot{\mathbf{u}}_\tau$ denotes the tangential velocity, q_τ is a given positive function and p_τ is a prescribed vector-valued function.

Let $T > 0$ and let $[0, T]$ be the time interval of interest. Under the previous assumptions, the classical formulation of the quasistatic problem for frictional adhesive contact problem between two electro-viscoelastic bodies with damage and normal damped response is the following.

Problem P

For l=1,2, determine a displacement field $\mathbf{u}^l : \Omega^l \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^l : \Omega^l \times [0,T] \to \mathbb{S}^d$, an electric potential field $\varphi^l : \Omega^l \times [0,T] \to \mathbb{S}^d$, an electric displacement $\boldsymbol{D}^l : \Omega^l \times [0,T] \to \mathbb{R}^d$, a bounding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ and a damage field $\alpha^l : \Omega^l \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}^l = \mathcal{A}^l \varepsilon(\dot{\mathbf{u}}^l) + \mathcal{B}^l\big(\varepsilon(\mathbf{u}^l), \alpha^l\big) - (\mathcal{E}^l)^* E(\varphi^l) \qquad \text{in } \Omega^l \times (0,T), \tag{2.3.1}$$

$$\boldsymbol{D}^l = \mathcal{E}^l \varepsilon(\mathbf{u}^l) + \mathcal{C}^l E(\varphi^l) \qquad \text{in } \Omega^l \times (0,T), \tag{2.3.2}$$

$$\operatorname{Div} \boldsymbol{\sigma}^l + \boldsymbol{f}_0^l = 0 \qquad \text{in } \Omega^l \times (0,T), \tag{2.3.3}$$

$$\operatorname{div} \boldsymbol{D}^l - q_0^l = 0 \qquad \text{in } \Omega^l \times (0,T), \tag{2.3.4}$$

$$\dot{\alpha}^l - \kappa^l \Delta \alpha^l + \partial \varphi_{\mathcal{K}^l}(\alpha^l) \ni \phi^l\big(\varepsilon(\mathbf{u}^l), \alpha^l\big) \qquad \text{in } \Omega^l \times (0,T), \tag{2.3.5}$$

$$\mathbf{u}^l = 0 \quad \text{on } \Gamma_1^l \times (0,T), \qquad \boldsymbol{\sigma}^l \boldsymbol{\nu}^l = \boldsymbol{f}_2^l \qquad \text{on } \Gamma_2^l \times (0,T), \tag{2.3.6}$$

$$\varphi^l = 0 \quad \text{on } \Gamma_a^l \times (0,T), \qquad \boldsymbol{D}^l.\boldsymbol{\nu}^l = q_2^l \qquad \text{on } \Gamma_b^l \times (0,T), \tag{2.3.7}$$

$$\sigma_\nu^1 = \sigma_\nu^1 \equiv \sigma_\nu, \text{ where } -\sigma_\nu = p_\nu([\dot{u}_\nu]) - \gamma_\nu \beta^2 R_\nu([u_\nu]) \qquad \text{on } \Gamma_3 \times (0,T) \tag{2.3.8}$$

$$\boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^1 \equiv \boldsymbol{\sigma}_\tau, \text{ where } -\boldsymbol{\sigma}_\tau = p_\tau([\dot{\mathbf{u}}_\tau]) + q_\tau(\beta) \boldsymbol{R}_\tau([\mathbf{u}_\tau]) \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.3.9}$$

$$\dot{\beta} = -\Big(\beta\big(\gamma_\nu (R_\nu([u_\nu]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_\tau])\right|^2\big) - \epsilon_a\Big)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.3.10}$$

$$\frac{\partial \alpha^l}{\partial \nu} = 0 \qquad \text{on } \Gamma^l \times (0,T), \tag{2.3.11}$$

$$\mathbf{u}^l(0) = \mathbf{u}_0^l, \qquad \alpha^l(0) = \alpha_0^l \qquad \text{in } \Omega^l, \tag{2.3.12}$$

$$\beta(0) = \beta_0 \qquad \text{in } \Gamma_3. \tag{2.3.13}$$

First, equations (2.3.1) and (2.3.2) represent the electro-viscoelastic constitutive law with damage, $\mathcal{A}^l$ and $\mathcal{B}^l$ are the viscosity and the elasticity operators, respectively, and $\mathcal{C}^l$ denotes the electric permitivity tensor, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposition. The equilibrium equations for the stress and electric-displacement fields, respectively, are represented by the relations (2.3.3) and (2.3.4). The evolution of the damage field is described in inclusion (2.3.5), where $\mathcal{K}^l$ signifies the set of acceptable damage functions established by

$$\mathcal{K}^l = \left\{ \zeta \in H^1(\Omega^l);\ \zeta \in [0,1] \text{ a.e. in } \Omega^l \right\},$$

the indicator function of the set $\mathcal{K}^l$ and $\partial\varphi_{\mathcal{K}^l}$ denotes its subdifferential. κ^l stands for the microcrack diffusion coefficient, which is assumed to be positive, ϕ^l is a specified constitutive function that identifies the sources of the damage in the system. Equations in (2.3.6) are the displacement and traction boundary conditions, while equations in (2.3.7) represent the electric boundary conditions.

The normal damped response condition with adhesion and its corresponding friction law are represented by (2.3.8) and (2.3.9) on the contact surface Γ_3. Next, equation (2.3.10) represents the ordinary differential equation that explains the evolution of the bonding field with given positive material parameters γ_ν, γ_τ and ϵ_a, in which $r_+ = max\{r, 0\}$. We observe that the adhesive process is irreversible and, in fact, that once debonding takes place, bonding cannot be restored because $\dot{\beta} \leq 0$. A homogeneous Neumann boundary condition for the damage field on Γ^l is represented by the relation (2.3.11). (2.3.12) represents the initial displacement field and the initial damage field. Finally (2.3.13) represents the initial condition in which β_0 is the given initial bonding field.

The following spaces are required in order to proceed with the variational formulation of the problem (2.3.1)–(2.3.13):

$$H^l = L^2(\Omega^l)^d = \left\{ \mathbf{u}^l = (u_i^l);\ u_i^l \in L^2(\Omega^l) \right\}, \qquad H_1^l = \left\{ \mathbf{u}^l \in H^l;\ \varepsilon(\mathbf{u}^l) \in \mathcal{H}^l \right\},$$
$$\mathcal{H}^l = \left\{ \boldsymbol{\tau}^l = (\tau_{ij}^l);\ \tau_{ij}^l = \tau_{ji}^l \in L^2(\Omega^l) \right\}, \qquad \mathcal{H}_1^l = \left\{ \boldsymbol{\tau}^l \in \mathcal{H}^l;\ \tau_{ij,j}^l \in H^l \right\}.$$

These are real Hilbert spaces that are provided with the canonical inner products that are provided by

$$(\mathbf{u}^l, \mathbf{v}^l)_{H^l} = \int_{\Omega^l} \mathbf{u}^l.\mathbf{v}^l\, dx, \qquad (\mathbf{u}^l, \mathbf{v}^l)_{H_1^l} = \int_{\Omega^l} \mathbf{u}^l.\mathbf{v}^l\, dx + \int_{\Omega^l} \nabla\mathbf{u}^l.\nabla\mathbf{v}^l\, dx,$$
$$(\boldsymbol{\sigma}^l, \boldsymbol{\tau}^l)_{\mathcal{H}^l} = \int_{\Omega^l} \boldsymbol{\sigma}^l.\boldsymbol{\tau}^l\, dx, \qquad (\boldsymbol{\sigma}^l, \boldsymbol{\tau}^l)_{\mathcal{H}_1^l} = \int_{\Omega^l} \boldsymbol{\sigma}^l.\boldsymbol{\tau}^l\, dx + \int_{\Omega^l} \operatorname{Div}\boldsymbol{\sigma}^l.\operatorname{Div}\boldsymbol{\tau}^l\, dx,$$

and the associated norms $|\cdot|_{H^l}$, $|\cdot|_{\mathcal{H}^l}$, $|\cdot|_{H_1^l}$ and $|\cdot|_{\mathcal{H}_1^l}$, respectively. Then we use the notation

$$\varepsilon(\mathbf{u}^l) = (\varepsilon_{ij}(\mathbf{u}^l)), \quad \varepsilon_{ij}(\mathbf{u}^l) = \frac{1}{2}(u^l_{i,j} + u^l_{j,i}), \quad \nabla\mathbf{u}^l = (u^l_{i,j}) \quad \forall\mathbf{u}^l \in \mathcal{H}_1^l,$$
$$\operatorname{Div}\boldsymbol{\sigma}^l = (\sigma^l_{ij,j}) \quad \forall\boldsymbol{\sigma}^l \in H_1^l.$$

For every element $\mathbf{u}^l \in H_1^l$ we denote by $\mathbf{u}^l$ its trace on Γ^l and by u^l_ν and $\mathbf{u}^l_\tau$ its normal and the tangential components on Γ^l given by

$$u^l_\nu = \mathbf{u}^l \cdot \boldsymbol{\nu}^l, \qquad \mathbf{u}^l_\tau = \mathbf{u}^l - u^l_\nu\boldsymbol{\nu}^l.$$

Similarly, for a regular tensor field $\boldsymbol{\sigma}^l : \Omega^l \to S^d$ we define its normal and tangential components on $\mathcal{H}_1^l$ by

$$\sigma^l_\nu = (\boldsymbol{\sigma}^l\boldsymbol{\nu}^l) \cdot \boldsymbol{\nu}^l, \qquad \boldsymbol{\sigma}^l_\tau = \boldsymbol{\sigma}^l\boldsymbol{\nu}^l - \sigma^l_\nu\boldsymbol{\nu}^l,$$

and the following Green's type formula holds:

$$(\boldsymbol{\sigma}^l, \varepsilon(\mathbf{u}^l))_{\mathcal{H}^l} + (\operatorname{Div}\boldsymbol{\sigma}^l, \mathbf{u}^l)_{H^l} = \int_{\Gamma^l} \boldsymbol{\sigma}^l\boldsymbol{\nu}^l \cdot \mathbf{u}^l \, da \quad \forall\mathbf{u}^l \in H_1^l. \tag{2.3.14}$$

To obtain the variational formulation of problem P we introduce the set of admissible bounding field, defined by

$$\mathcal{Z} = \left\{\varsigma \in L^\infty\big(0, T; L^2(\Gamma_3)\big);\ \varsigma \in [0.1] \text{ a.e. in } \Omega^l\right\},$$

and we need the following spaces

$$V^l = \left\{\mathbf{v}^l \in H_1^l;\ \mathbf{v}^l = 0 \,\text{on}\, \Gamma_1^l\right\}, \qquad W^l = \left\{\psi^l \in H^1(\Omega^l);\ \psi^l = 0 \,\text{on}\, \Gamma_a^l\right\},$$
$$\mathcal{W}^l = \left\{\boldsymbol{D}^l = (D_i^l);\ D_i^l \in L^2(\Omega^l),\ \operatorname{div}\boldsymbol{D}^l \in L^2(\Omega^l)\right\}.$$

The spaces V^l, W^l and $\mathcal{W}^l$ are reel Hilbert spaces with the inner products given by

$$(\mathbf{u}^l,\mathbf{v}^l)_{V^l} = \varepsilon(\mathbf{u}^l, \varepsilon(\mathbf{v}^l))_{\mathcal{H}^l}, \qquad (\varphi^l, \psi^l)_{W^l} = \int_{\Omega^l} \nabla\varphi^l.\nabla\psi^l \, dx,$$
$$(\boldsymbol{D}^l, \boldsymbol{E}^l)_{\mathcal{W}^l} = \int_{\Omega^l} \boldsymbol{D}^l.\boldsymbol{E}^l \, dx + \int_{\Omega^l} \operatorname{div} \boldsymbol{D}^l . \operatorname{div} \boldsymbol{E}^l \, dx,$$

where $\operatorname{div} \boldsymbol{D}^l = (\boldsymbol{D}^l_{i,i})$, and the associated norms $|\cdot|_{V^l}$, $|\cdot|_{W^l}$, and $|\cdot|_{\mathcal{W}^l}$, respectively. Notice that, since $meas(\Gamma^l_1) \geq 0$, the following Korn's inequality holds:

$$\left|\varepsilon(\mathbf{u}^l)\right|_{\mathcal{H}^l} \geq c_K \left|\mathbf{u}^l\right|_{H^l_1} \quad \forall\, \mathbf{u}^l \in V^l, \tag{2.3.15}$$

where c_K stands for a positive constant that is dependent only on Ω^l and Γ^l_1 (see [23]), and since $meas(\Gamma^l_a) \geq 0$, the following Friedrichs-Poincare inequality holds:

$$\left|\nabla\psi^l\right|_{L^2(\Omega^l)^d} \geq c_F \left|\psi^l\right|_{H^1(\Omega^l)} \quad \forall\, \psi^l \in W^l, \tag{2.3.16}$$

where c_F is a positive constant depending only on Ω^l and Γ^l_a. Additionally, according to the Sobolev trace theorem, two positive constants, c_0 and $\tilde{c}_0$, exist such that

$$\left|\mathbf{u}^l\right|_{L^2(\Gamma_3)^d} \leq c_0 \left|\mathbf{u}^l\right|_{V^l} \quad \forall\, \mathbf{u}^l \in V^l, \tag{2.3.17}$$
$$\left|\psi^l\right|_{L^2(\Omega^l)} \leq \tilde{c}_0 \left|\psi^l\right|_{W^l} \quad \forall\, \psi^l \in W^l. \tag{2.3.18}$$

Moreover, when $\boldsymbol{D}^l \in \mathcal{W}^l$ is a regular function, the following formula of Green's type holds

$$(\boldsymbol{D}^l, \nabla\psi^l)_{H^l} + (\operatorname{div} \boldsymbol{D}^l, \psi^l)_{L^2(\Omega^l)} = \int_{\Gamma^l} \boldsymbol{D}^l.\boldsymbol{\nu}^l\psi^l \, da \quad \forall\psi^l \in H^1(\Omega^l). \tag{2.3.19}$$

We define the product spaces to make the notations more explicit.

$$H = H^1 \times H^2, \quad \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \quad H_1 = H^1_1 \times H^2_1, \quad \mathcal{H}_1 = \mathcal{H}^1_1 \times \mathcal{H}^2_1,$$
$$V = V^1 \times V^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$
$$M_0 = L^2(\Omega^1) \times L^2(\Omega^2), \quad M_1 = H^1(\Omega^1) \times H^1(\Omega^2), \quad \mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2.$$

We take into account the following suppositions when studying the mechanical problem P The elasticity operator $\mathcal{B}^l$ and The viscosity operator $\mathcal{A}^l$ satisfy

$A(1):(a)\,\mathcal{A}^l:\Omega^l\times\mathbb{S}^d\to\mathbb{S}^d$

(b) There is $L_{\mathcal{A}^l}>0$ such that

$$\left|\mathcal{A}^l(x,\varepsilon_1)-\mathcal{A}^l(x,\varepsilon_2)\right|\le L_{\mathcal{A}^l}\left|\varepsilon_1-\varepsilon_2\right|\quad\forall\,\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\text{ a.e. }x\in\Omega^l.$$

(c) There exists $m_{\mathcal{A}^l}>0$ such that

$$(\mathcal{A}^l(x,\varepsilon_1)-\mathcal{A}^l(x,\varepsilon_2))\cdot(\varepsilon_1-\varepsilon_2)\ge m_{\mathcal{A}^l}\left|\varepsilon_1-\varepsilon_2\right|^2,\quad\forall\,\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ a.e.\ x\in\Omega^l.$$

(d) $x\mapsto\mathcal{A}^l(x,\varepsilon)$ is a Lebesgue measurable mapping on Ω^l, $\quad\forall\,\varepsilon\in\mathbb{S}^d$.

(e) $x\mapsto\mathcal{A}^l(x,0)$ is a map belonging to $\mathcal{H}^l$.

$A(2):(a)\,\mathcal{B}^l:\Omega^l\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$

(b) There is $L_{\mathcal{B}^l}>0$ such that

$$\left|\mathcal{B}^l(x,\varepsilon_1,\alpha_1)-\mathcal{B}^l(x,\varepsilon_2,\alpha_2)\right|\le L_{\mathcal{B}^l}\left|\varepsilon_1-\varepsilon_2\right|+\left|\alpha_1-\alpha_2\right|$$

$$\forall\,\varepsilon_1,\varepsilon_2\in\mathbb{S}^d,\ \alpha_1,\alpha_2\in\mathbb{R},\text{ a.e. }x\in\Omega^l.$$

(c) $x\mapsto\mathcal{B}^l(x,\varepsilon,\alpha)$ is Lebesgue measurable mapping on Ω^l, $\quad\forall\,\varepsilon\in\mathbb{S}^d$.

(d) $x\mapsto\mathcal{B}^l(x,0,0)$ is a map belonging to $\mathcal{H}^l$.

The electric permitivity tensor $\mathcal{C}^l$ and The piezoelectric tensor $\mathcal{E}^l$ satisfy

$A(3):(a)\,\mathcal{E}^l=(e^l_{ijk}):\Omega^l\times\mathbb{S}^d\to\mathbb{R}^d$

$(b)\,e^l_{ijk}=e^l_{ikj}\in L^\infty(\Omega^l),\ 1\le i,j,k\le d.$

$A(4):(a)\,\mathcal{C}^l=(c^l_{ij}):\Omega^l\times\mathbb{R}^d\to\mathbb{R}^d$

$(b)\,c^l_{ij}=e^l_{ji}\in L^\infty(\Omega^l),\ 1\le i,j\le d$

(c) There is $m_{\mathcal{C}^l}>0$ such that

$$c^l_{ij}(x)E_iE_j\ge m_{\mathcal{C}^l}\left|\boldsymbol{E}\right|^2\text{ for any }\boldsymbol{E}=(E_i)\in\mathbb{R}^d,\text{a.e. }x\in\Omega^l.$$

The damage source function ϕ^l satisfies

$A(5) : (a)\ \phi^l : \Omega^l \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$

(b) There is $L_{\phi^l} > 0$ such that

$$\left|\phi^l(x, \varepsilon_1, \alpha_1) - \phi^l(x, \varepsilon_2, \alpha_2)\right| \le L_{\phi^l} \left|\varepsilon_1 - \varepsilon_2\right| + \left|\alpha_1 - \alpha_2\right|$$

$\forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d$, $\forall\, \alpha_1, \alpha_2 \in \mathbb{R}$, a.e. $x \in \Omega^l$.

(c) $x \mapsto \phi^l(x, \varepsilon, \alpha)$ is Lebesgue measurable mapping on Ω^l,

for any $\varepsilon \in \mathbb{S}^d$ and $\alpha \in \mathbb{R}$

(d) $x \mapsto \phi^l(x, 0, 0)$ is a map belongs to $L^2(\Omega^l)$.

The normal damped response functions p_r $(r = \nu, \tau)$ satisfy

$A(6) : (a)\ p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) There is a constant $L_r > 0$ such that

$$|p_r(x, u_1) - p_r(x, u_2)| \le L_r\, |u_1 - u_2| \quad \forall\, u_1, u_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3.$$

(c) $\big(p_r(x, u_1) - p_r(x, u_2)\big)(u_1 - u_2) \ge 0 \quad \forall\, u_1, u_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3$

(d) $x \mapsto p_r(x, u)$ is measurable mapping on Γ_3, $\quad \forall\, u \in \mathbb{R}$.

(e) $p_r(x, u) = 0$ for all $u \le 0,\ a.e.\ x \in \Gamma_3$.

The tangential contact function q_τ satisfies

$A(7) : (a)\ q_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$

(b) There is a constant $L_\tau > 0$ such that

$$|q_\tau(x, u_1) - q_\tau(x, u_2)| \le L_\tau\, |u_1 - u_2| \quad \forall\, u_1, u_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3.$$

(c) There is a constant $M_\tau > 0$ such that

$$|q_\tau(x, u)| \le M_\tau \quad \forall\, u \in \mathbb{R},\ a.e.\ x \in \Gamma_3.$$

(d) $x \mapsto q_\tau(x, u)$ is measurable mapping on Γ_3, $\quad \forall\, u \in \mathbb{R}$.

(e) $x \mapsto q_\tau(x, 0) \in L^2(\Gamma_3)$, We assume that the densities, tractions and the forces satisfy

$$
\begin{aligned}
A(8):(a)\ f_0^l \in L^2(0,T;L^2(\Omega^l)^d), \qquad & (c)\ f_2^l \in L^2(0,T;L^2(\Gamma_2^l)^d)\\
(b)\ q_0^l \in C(0,T;L^2(\Omega^l)), \qquad & (d)\ q_2^l \in C(0,T;L^2(\Gamma_b^l)).
\end{aligned}
$$

Finally we assume that

$A(9):(a)$ The adhesion coefficients and the limit bound satisfy:

$$\epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \geq 0, \text{ a.e on } \Gamma_3.$$

(b) The microcrack diffusion coefficient verifies: $\kappa^l > 0$.

(c) The initial data satisfy:

$$\alpha_0^l \in \mathcal{K}^l, \quad \mathbf{u}_0^l \in V^l, \quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1, \text{ a.e. on } \Gamma_3.$$

Using the Riesz representation theorem, we have the linear mappings $q = (q^1, q^2) : [0,T] \to W$ and $\boldsymbol{f} = (\boldsymbol{f}^1, \boldsymbol{f}^2) : [0,T] \to V$ defined by:

$$(\boldsymbol{f}(t), \mathbf{v})_V = \sum_{l=1}^{2} \int_{\Omega^l} \boldsymbol{f}_0^l(t).\mathbf{v}^l \, dx + \sum_{l=1}^{2} \int_{\Gamma_2^l} \boldsymbol{f}_2(t).\mathbf{v}^l \, da \quad \forall\, \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V. \tag{2.3.20}$$

$$(q(t), \psi)_W = \sum_{l=1}^{2} \int_{\Omega^l} q_0^l(t)\psi^l \, dx + \sum_{l=1}^{2} \int_{\Gamma_b^l} q_2(t)\psi^l \, da \quad \forall\, \psi = (\psi^1, \psi^2) \in W. \tag{2.3.21}$$

The bilinear form a : is given by

$$a(\varphi, \xi) = \sum_{l=1}^{2} \kappa^l \int_{\Omega^l} \nabla\varphi^l.\nabla\xi^l d\,x \quad \forall\, \varphi = (\varphi^1, \varphi^2),\ \xi = (\xi^1, \xi^2) \in M_1. \tag{2.3.22}$$

The adhesion functional is denoted by $j_d : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ and defined by

$$j_d(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu([u_\nu])[v_\nu] + q_\tau(\beta)\boldsymbol{R}_\tau([\mathbf{u}_\tau]) \cdot [\mathbf{v}_\tau] \big) \, d\,a. \tag{2.3.23}$$

Also, we define the normal damped response functional

$$j : V \times V \to \mathbb{R}, \quad j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \big(p_\nu([u_\nu])[v_\nu] + p_\tau([\mathbf{u}_\tau]) \cdot [\mathbf{v}_\tau] \big)\, d\,a. \tag{2.3.24}$$

we observe that the conditions A(8) lead to

$$\boldsymbol{f} \in L^2(0, T; V), \quad q \in C(0, T; W). \tag{2.3.25}$$

2.3.2 Variational Formulation

Using Green's formulas, We can obtain the variational formulation of the frictional contact between two viscoelastic piezoelectric bodies with adhesion and damage, as follows.

Problem PV. Determine an electric potential field $\varphi = (\varphi^1, \varphi^2) : [0, T] \to W$, a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0, T] \to V$, a damage field $\alpha = (\alpha^1, \alpha^2) : [0, T] \to M_1$ and a bounding field $\beta : [0, T] \to L^\infty(\Gamma_3)$ such that

$$\begin{aligned} &\sum_{l=1}^{2} (\mathcal{A}^l \varepsilon(\dot{\mathbf{u}}^l(t)), \varepsilon(\mathbf{v}^l))_{\mathcal{H}^l} + \sum_{l=1}^{2} \big(\mathcal{B}^l(\varepsilon(\mathbf{u}^l(t)), \alpha^l(t)), \varepsilon(\mathbf{v}^l) \big)_{\mathcal{H}^l} \\ &+ \sum_{l=1}^{2} \big((\mathcal{E}^l)^* \nabla \varphi^l(t), \varepsilon(\mathbf{v}^l) \big)_{\mathcal{H}^l} + j_d(\beta(t), \mathbf{u}(t), \mathbf{v}) + j(\dot{\mathbf{u}}(t), \mathbf{v}) = (\boldsymbol{f}(t), \mathbf{v})_V \\ &\qquad \forall\, \mathbf{v} \in V, \ \text{ a.e. } t \in (0, T)\,, \end{aligned} \tag{2.3.26}$$

$$\begin{aligned} &\sum_{l=1}^{2} (\mathcal{C}^l \nabla \varphi^l(t), \nabla \psi^l)_{H^l} - \sum_{l=1}^{2} (\mathcal{E}^l \varepsilon(\mathbf{u}^l(t)), \nabla \psi^l)_{H^l} = (q(t), \psi)_W \\ &\qquad \forall\, \psi \in W, \ \text{ a.e. } t \in (0, T)\,, \end{aligned} \tag{2.3.27}$$

$$\begin{aligned} &\alpha(t) \in \mathcal{K}, \quad \sum_{l=1}^{2} (\dot{\alpha}^l(t), \xi^l - \alpha^l(t))_{L^2(\Omega^l)} + a(\alpha(t), \xi - \alpha(t)) \geq \\ &\sum_{l=1}^{2} \big(\phi^l(\varepsilon(\mathbf{u}^l(t)), \alpha^l(t)), \xi^l - \alpha^l(t) \big)_{L^2(\Omega^l)} \quad \forall\, \xi \in \mathcal{K}, \ \text{ a.e. } \ t \in (0, T)\,, \end{aligned} \tag{2.3.28}$$

$$\begin{aligned} &\dot{\beta}(t) = -\Big(\beta(t) \big(\gamma_\nu (R_\nu([u_\nu(t)]))^2 + \gamma_\tau \, |\boldsymbol{R}_\tau([\mathbf{u}_\tau(t)])|^2 \big) - \epsilon_a \Big)_+ \\ &\qquad \text{a.e. } \ t \in (0, T)\,, \quad \beta(t) \in \mathcal{Z}, \end{aligned} \tag{2.3.29}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \alpha(0) = \alpha_0, \qquad \beta(0) = \beta_0. \tag{2.3.30}$$

We will now consider this remark that is made below, which appears in many places in this subsection

Remark 2.3.1 *We observe that the limitation* $0 \leq \beta \leq 1$ *does not need to be imposed explicitly in the problems P and PV.*

Since equation (2.3.29) ensures that $\beta(x,t) \leq \beta_0(x)$ *, assumption* $\beta_0 \in \mathcal{Z}$ *in A(9)(c) demonstrates that* $\beta(x,t) \leq 1$ *for* $t \geq 0$, *a.e.* $x \in \Gamma_3$.

On the other hand, if (x, t0) = 0 at time t0, then (2.3.29) dictates that (x, t) = 0 for all t t0 and, thus, (x, t) = 0 for all t 0, i.e. x 3.

On the other hand, if $\beta(x,t_0) = 0$ *at time* t_0*, then it (2.3.29) dictates that* $\dot{\beta}(x,t) = 0$ *for all* $t \geq t_0$ *and thus,* $\beta(x,t) = 0$ *for all* $t \geq 0$, *a.e.* $x \in \Gamma_3$. *We come to the conclusion that* $0 \leq \beta(x,t) \leq 1$ *for every* $t \in [0,T]$, *a.e.* $x \in \Gamma_3$.

2.3.3 Existence and Uniqueness Results

The following is our main conclusion, which we assert here and support in the paragraph after.

Theorem 2.3.1 *Suppose that A(1)-A(9) hold. Then there is a unique solution* $\{\mathbf{u}, \varphi, \alpha, \beta\}$ *to Problem* PV*. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;V), \tag{2.3.31}$$

$$\varphi \in C(0,T;W), \tag{2.3.32}$$

$$\alpha \in W^{1,2}(0,T;M_0) \cap L^2(0,T;M_1), \tag{2.3.33}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}. \tag{2.3.34}$$

The functions $\mathbf{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \alpha$ and β which satisfy (2.3.1)–(2.3.2) and (2.3.26)-(2.3.30) are called a weak solution to the frictional contact problem P. We conclude by Theorem 2.3.1 that, under the stated assumptions, the mechanical problem (2.3.1)-(2.3.13) has a unique week solution $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \alpha, \beta)$.

To precise the regularity of the week solution , we note that the constitutive relations (2.3.1) and (2.3.2), the assumptions A(1)-A(4) and the regularities (2.3.31)–(2.3.32) show that $\boldsymbol{\sigma} \in C(0,T;\mathcal{H}), \boldsymbol{D} \in C(0,T;H)$; moreover, using (2.2.12), (2.2.13) and notations (2.2.4), (2.2.5), we obtain

$$\operatorname{Div} \boldsymbol{\sigma}^l(t) = -\boldsymbol{f}_0^l(t), \quad \operatorname{div} \boldsymbol{D}^l(t) = q_0^l(t) \qquad \forall\, t \in [0,T]\,,\, l = 1,2.$$

It follows now from the regularities A(8) that $\operatorname{Div} \boldsymbol{\sigma}^l \in C(0,T;H^l)$ and $\operatorname{div} \boldsymbol{D}^l \in C(0,T;L^2(\Omega^l))$, which shows that

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{2.3.35}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{2.3.36}$$

We conclude that the week solution $(\mathbf{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \alpha, \beta)$ of the electro-viscoelastic contact problem with adhesion and damage has the regularity (2.3.31)-(2.3.36).

Proof of Theorem 2.3.1

Theorem 2.3.1 will be proved in steps using arguments from nonlinear equations with monotone operators and a standard result for parabolic variational inequalities coupled with fixed point arguments. To achieve this, we'll make the following assumptions: A(1)–A(9) hold; C represents a strictly positive constant that may depend on the problem's data but is independent of time, and whose value may vary from place to place.

The following auxiliary problem for the displacement field, where $\eta \in C(0,T;V)$is given, is taken into consideration in the first step

Problem PV_η^1**.** Determine a displacement field $\mathbf{u}_\eta = (\mathbf{u}_\eta^1, \mathbf{u}_\eta^2) : [0,T] \to V$ such that

$$\sum_{l=1}^{2} \big(\mathcal{A}^l \varepsilon(\dot{\mathbf{u}}_\eta^l(t)), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) + (\eta(t), \mathbf{v})_V = (f(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V \quad a.e.\ t \in (0,T), \tag{2.3.37}$$

$$\mathbf{u}_\eta^l(0) = \mathbf{u}_0^l$$

The study of Problem PV_η^1 produced the next result

Lemma 2.3.1 *Problem PV_η^1 has an unique solution, and it satisfies $\mathbf{u}_\eta \in C^1(0,T;V)$. Additionally, if $\mathbf{u}_{\eta_i}$ is the solution of these Problems PV_η^1 for $\eta_i \in L^2(0,T;V)$, $i = 1,2$ then there is $C > 0$ such that*

$$|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)|_V^2 \le C \int_0^t |\eta_1(s) - \eta_2(s)|_V^2 \, ds. \tag{2.3.38}$$

Proof. We use the Riesz's representation theorem to determine the element $f_\eta(t) \in V$ and the operator $T : V \to V$ by

$$(T\mathbf{u}, \mathbf{v})_V = \sum_{l=1}^{2} (\mathcal{A}^l \varepsilon(\mathbf{u}^l), \varepsilon(\mathbf{v}^l))_{\mathcal{H}^l} + j(\mathbf{u}, \mathbf{v}), \tag{2.3.39}$$

$$(f_\eta(t), \mathbf{v})_V = (f(t), \mathbf{v})_V - (\eta(t), \mathbf{v})_V, \tag{2.3.40}$$

for every $\mathbf{u}, \mathbf{v} \in V$, $t \in [0,T]$. Let $\mathbf{u}_1, \mathbf{u}_2 \in V$. Using (2.3.39) we obtain

$$\begin{aligned}(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V = \sum_{l=1}^{2} \big(\mathcal{A}^l \varepsilon(\mathbf{u}_1^l) - \mathcal{A}^l \varepsilon(\mathbf{u}_2^l), \varepsilon(\mathbf{u}_1^l) - \varepsilon(\mathbf{u}_2^l)\big)_{\mathcal{H}^l} \\ + j(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - j(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)\end{aligned} \tag{2.3.41}$$

Using (2.3.24) and A(6)(c) we obtain

$$\begin{aligned}j(\mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - j(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = \int_{\Gamma_3} \big(p_\nu([u_{1\nu}]) - p_\nu([u_{2\nu}])[u_{1\nu} - u_{2\nu}] \, da \\ + \int_{\Gamma_3} \big(p_\tau([\mathbf{u}_{1\tau}]) - p_\tau([\mathbf{u}_{2\tau}]) \cdot [\mathbf{u}_{1\tau} - \mathbf{u}_{12\tau}]\big) \, da \ge 0,\end{aligned} \tag{2.3.42}$$

And taking into account (2.3.41), (2.3.42) and A(1)(c), we get

$$(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V \geq m\,|\mathbf{u}_1 - \mathbf{u}_2|_V^2 \qquad \forall\,\mathbf{u}_1, \mathbf{u}_2 \in V, \tag{2.3.43}$$

where constant $m = \min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2}) > 0$. Using again (2.3.39) leads to

$$(T\mathbf{u}_1 - T\mathbf{u}_2, \mathbf{v})_V = \sum_{l=1}^{2} \big(\mathcal{A}^l\varepsilon(\mathbf{u}_1^l) - \mathcal{A}^l\varepsilon(\mathbf{u}_2^l), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + j(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) \tag{2.3.44}$$

for all $\mathbf{v} \in V$ and, by (2.3.24) and A(6)(b) we find

$$\begin{aligned} j(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) = &\int_{\Gamma_3} \big(p_\nu([u_{1\nu}]) - p_\nu([u_{2\nu}])[v_\nu]\,da \\ &+ \int_{\Gamma_3} \big(p_\tau([\mathbf{u}_{1\tau}]) - p_\tau([\mathbf{u}_{2\tau}]) \cdot [\mathbf{v}_\tau]\big)\,da \leq C\,|\mathbf{u}_1 - \mathbf{u}_2|_V \end{aligned} \tag{2.3.45}$$

and, keeping in mind (2.3.44), (2.3.45) and A(1)(b), we find

$$|T\mathbf{u}_1 - T\mathbf{u}_2|_V \leq (C + L_{\mathcal{A}})\,|\mathbf{u}_1 - \mathbf{u}_2|_V \quad \forall\,\mathbf{u}_1, \mathbf{u}_2 \in V, \tag{2.3.46}$$

where the positive constant $L_{\mathcal{A}} = \max(L_{\mathcal{A}^1}, L_{\mathcal{A}^2})$.

The operator $T : V \to V$ is strongly monotone , as demonstrated by inequality (2.3.43). Moreover, Inequality (2.3.46) also suggests that T is a Lipschitz continuous. According to a standard result for nonlinear equations (see, for example, [12]), there exists an unique element $\mathbf{v}_\eta \in C(0, T, V)$ that satisfies

$$T\mathbf{v}_\eta(t) = f_\eta(t) \quad a.e.\ t \in (0, t). \tag{2.3.47}$$

We consider the function $\mathbf{u}_\eta : [0, T] \to V$ defined by

$$\mathbf{u}_\eta = \int_0^t \mathbf{v}_\eta(s)ds + \mathbf{u}_0. \tag{2.3.48}$$

It follows from (2.3.39), (2.3.47)–(2.3.48) that $\mathbf{u}_\eta$ is a solution of the equatiom (2.3.37) and it satisfies $\mathbf{u}_\eta \in C^1(0, T; V)$. It remains to show estimate (2.3.38). Let $\eta_1, \eta_2 \in C(0, T, V)$ and use the notation $\mathbf{u}_i = \mathbf{u}_{\eta_i}$, $\mathbf{v}_i = \mathbf{v}_{\eta_i} = \dot{\mathbf{u}}_{\eta_i}$ for $i = 1, 2$. Moreover, using (2.3.37) we obtain

$$\sum_{l=1}^{2} \left(\mathcal{A}^l \varepsilon(\dot{\mathbf{u}}_i^l(t)), \varepsilon(\mathbf{v}^l)\right)_{\mathcal{H}^l} + j(\dot{\mathbf{u}}_i(t), \mathbf{v}) + (\eta_i(t), \mathbf{v})_V = (f(t), \mathbf{v})_V.$$

We Subtract the two obtained equations, by choosing $\mathbf{v} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$ as test function, to find

$$\sum_{l=1}^{2} \left(\mathcal{A}^l \varepsilon(\mathbf{v}_1^l) - \mathcal{A}^l \varepsilon(\mathbf{v}_2^l), \varepsilon(\mathbf{v}_1^l) - \varepsilon(\mathbf{v}_2^l)\right)_{\mathcal{H}^l} + j\left(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2\right) - j\left(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\right) \tag{2.3.49}$$

$$= \left(\eta_2 - \eta_1, \mathbf{v}_1 - \mathbf{v}_2\right)_V .$$

It follows from (2.3.24) and A(6) that

$$j\left(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2\right) - j\left(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\right) \geq 0. \tag{2.3.50}$$

Keeping in mind (2.3.17) and A(1)(c) we deduce that

$$\sum_{l=1}^{2} \left(\mathcal{A}^l \varepsilon(\mathbf{v}_1^l) - \mathcal{A}^l \varepsilon(\mathbf{v}_2^l), \varepsilon(\mathbf{v}_1^l) - \varepsilon(\mathbf{v}_2^l)\right)_{\mathcal{H}^l} \geq C \left|\mathbf{v}_1(t) - \mathbf{v}_2(t)\right|_V^2 \quad \forall t \in [0, T]\,. \tag{2.3.51}$$

The Cauchy-Schwartz inequality also allows us to obtain

$$\left(\eta_2(t) - \eta_1(t), \mathbf{v}_1(t) - \mathbf{v}_2(t)\right)_{V' \times V} \leq \left|\eta_2(t) - \eta_1(t)\right|_{V'} \left|\mathbf{v}_1(t) - \mathbf{v}_2(t)\right|_V . \tag{2.3.52}$$

We Combine (2.3.49)–(2.3.52) with some algebraic manipulations to obtain

$$\left|\mathbf{v}_1(t) - \mathbf{v}_2(t)\right|_V \leq C \left|\eta_1(t) - \eta_2(t)\right|_V .$$

Since $\mathbf{u}_i(t) = \int_0^t \mathbf{v}_i(s) ds + \mathbf{u}_0$ and $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$ we have

$$\left|\mathbf{u}_1(t) - \mathbf{u}_2(t)\right|_V \leq \int_0^t \left|\mathbf{v}_1(s) - \mathbf{v}_2(s)\right|_V ds.$$

From the two previous inequalities we deduce (3.2.39) , which concludes the proof. ■

Let $\eta \in C(0,T;V)$ be provided. The following intermediate variational problem is considered in the second step

Problem PV_η^2

Determine an electric potential field $\varphi_\eta = (\varphi_\eta^1, \varphi_\eta^2) : [0,T] \to W$ such that

$$\sum_{l=1}^{2}(\mathcal{C}^l \nabla \varphi_\eta^l(t), \nabla \psi^l)_{H^l} - \sum_{l=1}^{2}(\mathcal{E}^l \varepsilon(\mathbf{u}_\eta^l(t)), \nabla \psi^l)_{H^l} = (q(t), \psi)_W \tag{2.3.53}$$

$$\forall\, \psi \in W, \ \text{ a.e. } t \in (0,T)\,.$$

We have the following existence and uniqueness result.

Lemma 2.3.2 *Problem PV_η^2 has a unique solution φ_η which satisfies the regularity (2.3.32). Moreover, if φ_{η_i} represents the solution of problem (2.3.53) for $\eta_i \in C(0,T;V)$, $i = 1, 2$ then there exists $C > 0$ such that*

$$|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)|_V^2 \le C\,|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)|_V^2\, ds. \tag{2.3.54}$$

Proof. Let $h(.,.) : W \times W \to \mathbb{R}$ be a bilinear form defined by

$$h(\varphi, \psi) = \sum_{l=1}^{2}(\mathcal{C}^l \nabla \varphi^l(t), \nabla \psi^l)_{H^l}. \tag{2.3.55}$$

We demonstrate that the bilinear form $h(.,.)$ is continuous, symmetric, and coercive on W, using A(4) and the Friedrichs-Poincare inequality (2.3.16). Additionally, we may define an element $q_\eta : [0,T] \to W$ using the Riesz Representation Theorem, in order that

$$(q_\eta(t), \psi)_W = \sum_{l=1}^{2}(\mathcal{E}^l \varepsilon(\mathbf{u}_\eta^l(t)), \nabla \psi^l)_{H^l} + (q(t), \psi)_W \qquad \forall\, \phi \in W,\ t\, \in (0,T).$$

By using the Lax-Milgram Theorem, we may conclude there is only one element $\varphi_\eta(t) \in W$ such that

$$h(\varphi_\eta(t), \psi) = (q_\eta(t), \psi)_W \qquad \forall\, \phi \in W. \tag{2.3.56}$$

From (2.3.56) it is evident that φ_η is a solution of PV_η^2. Let $t_1, t_2 \in [0, T]$, using (2.3.18), (3.2.53) and A(4) we obtain

$$|\varphi_\eta(t_1) - \varphi_\eta(t_2)|_W \leq C\big(\, |\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |q(t_1) - q(t_2)|_W\,\big). \tag{2.3.57}$$

It follows from (2.3.57), the regularity of $\mathbf{u}_\eta$ and q that $\varphi_\eta \in C^1(0, T; W)$. It remains to show estimate (2.3.56). Let $\eta_1, \eta_2 \in C(0, T, V)$ and use the notation $\mathbf{u}_i = \mathbf{u}_{\eta_i}$, $\varphi_i = \varphi_{\eta_i}$ for $i = 1, 2$. Moreover, using (2.3.55) we obtain

$$\sum_{l=1}^{2} (\mathcal{C}^l \nabla \varphi_i^l(t), \nabla \psi^l)_{H^l} - \sum_{l=1}^{2} (\mathcal{E}^l \varepsilon(\mathbf{u}_i^l(t)), \nabla \psi^l)_{H^l} = (q(t), \psi)_W.$$

We subtract the two obtained equations, by choosing $\psi = \varphi_1 - \varphi_2$ as test function, to find

$$\sum_{l=1}^{2} (\mathcal{C}^l \nabla \varphi_1^l - \mathcal{C}^l \nabla \varphi_2^l, \nabla \varphi_1^l - \nabla \varphi_2^l)_{H^l} = \sum_{l=1}^{2} (\mathcal{E}^l \varepsilon(\mathbf{u}_1^l) - \mathcal{E}^l \varepsilon(\mathbf{u}_2^l), \nabla \varphi_1^l - \nabla \varphi_2^l)_{H^l}.$$

Keeping in mind (2.3.18), the properties of the operators $\mathcal{C}^l, \mathcal{E}^l$ and using Cauchy-Schwartz inequality we find

$$|\varphi_1(t) - \varphi_2(t)|_V^2 \leq C\, |\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 \, ds.$$

■

In the third step, we construct the following Cauchy problem for the bonding field.

Problem PV_η^3

Determine a bonding field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}(t) = -\Big(\beta_\eta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t)])\right|^2\big) - \epsilon_a\Big)_+ \tag{2.3.58}$$

$$\beta_\eta(0) = \beta(0). \tag{2.3.59}$$

We have the following result

Lemma 2.3.3 *The Problem PV_η^3 has an unique solution , and it satisfies the regularity given in (2.3.34). Additionally, if β_i is the solution of these problems $PV_{\eta_i}^3$ for $\eta_i \in C(0,T;V)$, $i = 1,2$ then there is $C > 0$ such that*

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)} \le C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V \, ds. \quad \forall\, t \in [0,T]\,. \tag{2.3.60}$$

Proof. Consider the mapping $H_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$H_\eta(t,\beta) = -\Big(\beta(t)\big(\gamma_\nu(R_\nu([u_{\eta\nu}(t)]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t)])\right|^2\big) - \epsilon_a\Big)_+ ,$$

for each $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. The characteristics of the truncation operator R_ν and $\boldsymbol{R}_\tau$ imply that H_η is Lipschitz continuous with respect to the second variable, uniformly in time. In addition, the mapping $t \to H_\eta(t,\beta)$ for any $\beta \in L^2(\Gamma_3)$, belongs to $L^\infty(0,T,L^2(\Gamma_3))$. Therefore using a version of Cauchy-Lipschitz theorem (see, for example, [54, p.48]) leads us to conclude that there exists an unique function $\beta_\eta \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ that satisfies (2.3.58) and (2.3.59). Additionally, the justifications provided in Remark 1 demonstrate that $0 \le \beta_\eta(t) \le 1$ for every $t \in [0,T]$, a.e on Γ_3. As a result, we discover that $\beta_\eta \in \mathcal{Z}$, from the definition of the set $\mathcal{Z}$, which completes the demonstration of the first part of the lemma.

On the other hand, let $t \in [0,T]$ and $\mathbf{u}_1, \mathbf{u}_2 \in C(0,T,V)$. We from the Cauchy problem (2.3.58)–(2.3.59)

$$\beta_i(t) = \beta_0 - \int_0^t \Big(\beta_i(s)\big(\gamma_\nu(R_\nu([u_{i\nu}(s)]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_{i\tau}(s)])\right|^2\big) - \epsilon_a\Big)_+ ds,$$

and then

$$\begin{aligned} |\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \leq C \int_0^t \Big|\beta_1(s)\big(R_\nu([u_{1\nu}(s)])\big)^2 - \beta_2(s)\big(R_\nu([u_{2\nu}(s)])\big)^2\Big|_{L^2(\Gamma_3)} ds \\ + C \int_0^t \Big|\big(\boldsymbol{R}_\tau([\mathbf{u}_{1\tau}(s)])\big)^2 - \big(\boldsymbol{R}_\tau([\mathbf{u}_{2\tau}(s)])\big)^2\Big|_{L^2(\Gamma_3)} ds. \end{aligned}$$

We use the definition of the truncation operators R_ν and $\boldsymbol{R}_\tau$, and write $\beta_1 = \beta_1 - \beta_2 + \beta_2$ to obtain

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \leq C\bigg(\int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)}\, ds + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{L^2(\Gamma_3)^d}\, ds\bigg).$$

Applying Gronwall's inequality leads to

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{L^2(\Gamma_3)^d}\, ds,$$

also, from the relation (1.2.26) we discover

$$|\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)} \leq C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds. \quad \forall\, t \in [0,T]\,.$$

■

Let $\mu \in C(0,T;M_0)$ be provided. The following variational problem for the damage field is considered in the next step

Problem PV_μ

Determine the damage field $\alpha_\mu : [0,T] \to M_1$ such that

$$\alpha_\mu(t) \in \mathcal{K}, \quad \sum_{l=1}^{2} \left(\dot{\alpha}^l_\mu(t), \xi^l - \alpha^l_\mu(t)\right)_{L^2(\Omega^l)} + a(\alpha_\mu(t), \xi - \alpha_\mu(t)) \tag{2.3.61}$$
$$\geq \sum_{l=1}^{2} \left(\mu^l(t), \xi^l - \alpha^l_\mu(t)\right)_{L^2(\Omega^l)} \quad \forall \xi \in \mathcal{K}, \ a.e. \ t \in (0,T)$$
$$\alpha_\mu(0) = \alpha_0. \tag{2.3.62}$$

We apply Theorem 1.3.7 to problem PV_μ.

Lemma 2.3.4 *The auxiliary Problem PV_μ has an unique solution α_μ, and it satisfies the regularity given in (2.3.33). Additionally, if α_i is the solution of these problems PV_{μ_i} for $\mu_i \in L^2(0,T;M_0)$, $i = 1,2$ then there is $C > 0$ such that*

$$|\alpha_1(t) - \alpha_2(t)|^2_{M_0} \leq C \int_0^t |\mu_1(s) - \mu_2(s)|^2_{M_0} \, ds. \quad \forall\, t \in [0,T]\,. \tag{2.3.63}$$

Proof. We denote by M_1' the dual space of M_1 and, identifying M_0' with M_0, the inclusion mapping of $\left(M_1, |\,.\,|_{M_1}\right)$ into $\left(M_0, |\,.\,|_{M_0}\right)$ is continuous and its range is dense. Thus we can write the Gelfand triple

$$M_1 \subset M_0 = M_0' \subset M_1'.$$

We denote by $(\,.,.\,)_{M_1'\times M_1}$ the duality pairing between ${M_1}'$ and M_1, we note that $\mathcal{K}$ is a closed convex set in M_1, thus we have

$$(\alpha, \xi)_{M_1'\times M_1} = (\alpha, \xi)_{M_0} \quad \forall\, \alpha \in M_0, \xi \in M_1.$$

Taking into consideration the fact that $\alpha_0 \in \mathcal{K}$ in $A(9)(c)$, using (2.3.22) the definition of the bilinear form a, $A(9)(b)$, it is easy to see that Lemma 2.3.4 is a direct consequence of Theorem 1.3.7.

Now we use the notation $\alpha_\mu = \alpha_i$ for $i = 1, 2.$ and let $\mu_1, \mu_2 \in C(0, T; M_0)$, we take the substitution $\mu = \mu_1$, $\mu = \mu_2$ in (2.3.61) and subtract the two obtained equations, we choose $\xi = \alpha_1 - \alpha_2$ as test function to find

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{M_0} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \leq (\mu_1 - \mu_2, \alpha_1 - \alpha_2)_{M_0} \quad a.e.\ t \in (0, T).$$

Taking into account that $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$, we obtain

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{M_0} \leq (\mu_1 - \mu_2, \alpha_1 - \alpha_2)_{M_0}. \tag{2.3.64}$$

Using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and integrating the previous inequality (2.3.64) with regard to time, allow us to obtain

$$\frac{1}{2} \left|\alpha_1(t) - \alpha_2(t)\right|^2_{M_0} \leq \int_0^t (\mu_1 - \mu_2, \beta_1 - \beta_2)_{M_0} ds.$$

Through the use of Hölder's and Young's inequalities, we deduce that

$$\left|\alpha_1(t) - \alpha_2(t)\right|^2_{M_0} \leq \int_0^t \left|\mu_1(s) - \mu_2(s)\right|^2_{M_0} ds + \int_0^t \left|\alpha_1(s) - \alpha_2(s)\right|^2_{M_0} ds.$$

Combining the previous inequality with Gronwall's inequality results in

$$\left|\alpha_1(t) - \alpha_2(t)\right|^2_{M_0} \leq C \int_0^t \left|\mu_1(s) - \mu_2(s)\right|^2_{M_0} ds \qquad \forall\, t \in [0, T]\,.$$

■

We now pass to the final step. For every $(\eta, \mu) \in C(0, T; V \times M_0)$ we denote by $\mathbf{u}_\eta, \varphi_\eta, \beta_\eta$ and α_μ the displacement, the potential electric, the bonding and the damage fields obtained in Lemmas 1,2,3,4 respectively. Using the properties of the operators $\mathcal{B}^l, \mathcal{E}^l$, the adhesion functional j_d and the function ϕ^l, for $t \in [0, T]$, we apply the Riesz representation theorem to define the function $\Lambda : t \to V \times M_0$ by

$$\Lambda(\eta,\mu)(t) = (\Lambda_1(\eta,\mu)(t), \Lambda_2(\eta,\mu)(t)), \tag{2.3.65}$$

$$(\Lambda_1(\eta,\mu)(t),\mathbf{v})_V = j_d(\beta_\eta(t),\mathbf{u}_\eta(t),\mathbf{v}) + \sum_{l=1}^{2}\big(\mathcal{B}^l(\varepsilon(\mathbf{u}_\eta^l(t)),\alpha_\mu^l(t)),\varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} \tag{2.3.66}$$
$$+\sum_{l=1}^{2}\big((\mathcal{E}^l)^*\nabla\varphi_\eta^l(t),\varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l}$$

$$\Lambda_2(\eta,\mu)(t) = (\phi^1(\varepsilon(\mathbf{u}_\eta{}^1(t)),\alpha_\mu^1(t)),\phi^2(\varepsilon(\mathbf{u}_\eta{}^2(t)),\alpha_\mu^2(t))). \tag{2.3.67}$$

We have the following result

Lemma 2.3.5 *The function $\Lambda : t \to V \times M_0$ is continuous, for each $(\eta,\mu) \in C(0,T;V\times M_0)$. Additionally, there is a a unique element $(\eta^*,\mu^*) \in C(0,T;V\times M_0)$ that $\Lambda(\eta^*,\mu^*) = (\eta^*,\mu^*)$*

Proof. Let $t_1, t_2 \in [0,T]$ and $(\eta,\mu) \in C(0,T;V\times M_0)$. Using (2.3.66) and (2.3.23) we obtain

$$|\Lambda_1(\eta,\mu)(t_1) - \Lambda_1(\eta,\mu)(t_2)|_V \le \sum_{l=0}^{1}\big|\mathcal{B}^l(\varepsilon(\mathbf{u}_\eta^l(t_1)),\alpha_\mu^l(t_1)) - \mathcal{B}^l(\varepsilon(\mathbf{u}_\eta^l(t_2)),\alpha_\mu^l(t_2))\big|_{\mathcal{H}^l}$$
$$+\sum_{l=0}^{1}\big|\mathcal{E}^l)^*\nabla\varphi_\eta^l(t_1) - \mathcal{E}^l)^*\nabla\varphi_\eta^l(t_2)\big|_{\mathcal{H}^l} \tag{2.3.68}$$
$$+C\big|\beta_\eta^2(t_1)R_\nu([u_{\eta\nu}(t_1)]) - \beta_\eta^2(t_2)R_\nu([u_{\eta\nu}(t_2)])\big|_{L^2(\Gamma_3)}$$
$$+C\big|q_\tau(\beta_\eta(t_1))\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t_1)]) - q_\tau(\beta_\eta(t_2))\boldsymbol{R}_\tau([\mathbf{u}_{\eta\tau}(t_2)])\big|_{L^2(\Gamma_3)}.$$

Keeping in mind the inequality $0 \le \beta_\eta \le 1$, the properties of the operators $\mathcal{B}^l, \mathcal{E}^l, R_\tau$ and $\boldsymbol{R}_\nu$ and (2.3.17), A(6)-(7), we find

$$|\Lambda_1(\eta,\mu)(t_1) - \Lambda_1(\eta,\mu)(t_2)|_V \le C\big(|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)|_V + |\varphi_\eta(t_1) - \varphi_\eta(t_2)|_W$$
$$+|\beta_\eta(t_1) - \beta_\eta(t_2)|_{L^2(\Gamma_3)} + |\alpha_\mu(t_1) - \alpha_\mu(t_2)|_{M_0}\big). \tag{2.3.69}$$

Next, we conclude from (2.3.69) that $\Lambda_1(\eta,\mu) \in C(0,T;V)$, based on the regularities of $\mathbf{u}_\eta, \varphi_\eta, \beta_\eta$ and α_μ expressed in (2.3.31), (2.3.32), (2.3.33) and (2.3.34), respectively.

A similar argument from (2.3.67) and A(5) leads to the conclusion that

$$|\Lambda_2(\eta,\mu)(t_1)-\Lambda_2(\eta,\mu)(t_2)|_{M_0}\le C\big(\,|\mathbf{u}_\eta(t_1)-\mathbf{u}_\eta(t_2)|_V+|\alpha_\mu(t_1)-\alpha_\mu(t_2)|_{M_0}\,\big). \quad (2.3.70)$$

Thus $\Lambda_2(\eta,\mu)\in C(0,T;M_0)$ and $\Lambda(\eta,\mu)\in C(0,T;V\times M_0)$.

Let now $(\eta_1,\mu_1),(\eta_2,\mu_2)\in C(0,T;V\times M_0)$ and denote $\mathbf{u}_i=\mathbf{u}_{\eta_i}$, $\mathbf{v}_i=\mathbf{v}_{\eta_i}=\dot{\mathbf{u}}_{\eta_i}$, $\varphi_i=\varphi_{\eta_i}$, $\beta_i=\beta_{\eta_i}$ and $\alpha_i=\alpha_{\mu_i}$ for $i=1,2$.
Arguments similar to those used in the proof of (2.3.69) and (2.3.70) yield

$$\begin{aligned}&|\Lambda(\eta_1,\mu_1)(t)-\Lambda(\eta_2,\mu_2)(t)|^2_{V\times M_0}\\ &\le C\big(\,|\mathbf{u}_1(t)-\mathbf{u}_2(t)|^2_V+|\varphi_1(t)-\varphi_2(t)|^2_W\\ &+|\beta_1(t)-\beta_2(t)|^2_{L^2(\Gamma_3)}+|\alpha_1(t)-\alpha_2(t)|^2_{M_0}\,\big).\end{aligned} \quad (2.3.71)$$

We substitute inequalities (2.3.38), (2.3.54), (2.3.60) and (2.3.63) in (2.3.71) to find

$$|\Lambda(\eta_1,\mu_1)(t)-\Lambda(\eta_2,\mu_2)(t)|^2_{V\times M_0}\le C\int_0^t|(\eta_1,\mu_1)(s)-(\eta_2,\mu_2)(s)|^2_{V\times M_0}\,ds.$$

If we repeat this inequality m times, we get

$$|\Lambda^m(\eta_1,\mu_1)-\Lambda^m(\eta_2,\mu_2)|^2_{C(0,T,V\times M_0)}\le\frac{C^mT^m}{m!}\,|(\eta_1,\mu_1)-(\eta_2,\mu_2)|^2_{C(0,T,V\times M_0)}\,.$$

As a result, for sufficiently large m, Λ^m is a contraction on the Banach space $C(0,T;V\times M_0)$, and therefore Λ has a unique fixed point (η^*,μ^*). ■

We now possess the necessary elements to demonstrate Theorem 2.3.1. **Proof of Theorem 2.3.1.** Let $(\eta^*,\mu^*)\in C(0,T;V\times M_0)$ be the fixed point of the operator Λ defined by (2.3.65)–(2.3.67) and let $\mathbf{u}_{\eta^*}$, φ_{η^*}, β_{η^*}, α_{μ^*} be the solutions of the four intermediate problems defined above. We denote

$$\mathbf{u}_*=\mathbf{u}_{\eta^*},\ \varphi_*=\varphi_{\eta^*},\ \beta_*=\beta_{\eta^*},\ \alpha_*=\alpha_{\eta^*}. \quad (2.3.72)$$

We consider the functions $\boldsymbol{\sigma}_*^l : [0,T] \to \mathcal{H}$ and $\boldsymbol{D}_*^l : [0,T] \to H$ defined by

$$\boldsymbol{\sigma}^l = \mathcal{A}^l \varepsilon(\dot{\mathbf{u}}^l) + \mathcal{B}^l\big(\varepsilon(\mathbf{u}^l), \alpha^l\big) + (\mathcal{E}^l)^* \nabla(\varphi^l) \quad l = 1, 2 \tag{2.3.73}$$

$$D^l = \mathcal{E}^l \varepsilon(\mathbf{u}^l) - \mathcal{C}^l \nabla(\varphi^l) \quad l = 1, 2. \tag{2.3.74}$$

we will demonstrate that the quadruple $\{\mathbf{u}_*, \varphi_*, \beta_*, \alpha_*\}$ is the only solution to Problem PV that satisfies the regularities (2.3.31)–(2.3.34).

Indeed, we write (2.3.37), (2.3.53), (2.3.58) and (2.3.61) for $\eta = \eta^*$, $\mu = \mu^*$ and use (3.2.76) to obtain

$$\sum_{l=1}^{2} \big(\mathcal{A}^l \varepsilon(\dot{\mathbf{u}}_*^l(t)), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + j(\dot{\mathbf{u}}_*(t), \mathbf{v}) + (\eta^*(t), \mathbf{v})_V = (f(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V \quad a.e.\ t \in (0,T), \tag{2.3.75}$$

$$\sum_{l=1}^{2} (\mathcal{C}^l \nabla \varphi_*^l(t), \nabla \psi^l)_{H^l} - \sum_{l=1}^{2} (\mathcal{E}^l \varepsilon(\mathbf{u}_*^l(t)), \nabla \psi^l)_{H^l} = (q(t), \psi)_W \quad \forall\, \psi \in W, \ \text{a.e. } t \in (0,T)\,, \tag{2.3.76}$$

$$\dot{\beta}(t) = -\bigg(\beta_*(t)\big(\gamma_\nu (R_\nu([u_{*\nu}(t)]))^2 + \gamma_\tau \left|\boldsymbol{R}_\tau([\mathbf{u}_{*\tau}(t)])\right|^2\big) - \epsilon_a\bigg)_+ \quad \text{a.e. } \ t \in [0,T]\,, \tag{2.3.77}$$

$$\alpha_*(t) \in \mathcal{K}, \quad \sum_{l=1}^{2} \big(\dot{\alpha}_*^l(t), \xi^l - \alpha_*^l(t)\big)_{L^2(\Omega^l)} + a(\alpha_*(t), \xi - \alpha_*(t)) \tag{2.3.78}$$
$$\geq \sum_{l=1}^{2} \big(\mu^{*l}(t), \xi^l - \alpha_*^l(t)\big)_{L^2(\Omega^l)} \quad \forall \xi \in \mathcal{K}, \ a.e.\ t \in [0,T]\,.$$

Using equality $\Lambda_1(\eta^*, \mu^*) = \eta^*$ combined with (2.3.66) leads to

$$j_d(\beta_*(t), \mathbf{u}_*(t), \mathbf{v}) + \sum_{l=1}^{2} \big(\mathcal{B}^l(\varepsilon(\mathbf{u}_*^l(t)), \alpha_\mu^l(t)), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + \sum_{l=1}^{2} \big((\mathcal{E}^l)^* \nabla \varphi_*^l(t), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} = (\eta^*(t), \mathbf{v})_V. \tag{2.3.79}$$

From the equality $\Lambda_2(\eta^*, \mu^*) = \mu^*$ and (2.3.67) we can write for $l = 1, 2$

$$\phi^l(\varepsilon(\mathbf{u_*}^1(t)), \alpha_*^l(t)) = \mu^{*l}. \tag{2.3.80}$$

We now substitute (2.3.79) in (3.2.79) to obtain

$$\begin{aligned}
&\sum_{l=1}^{2}(\mathcal{A}^l\varepsilon(\dot{\mathbf{u}}_*^l(t)), \varepsilon(\mathbf{v}^l))_{\mathcal{H}^l} + \sum_{l=1}^{2}\big(\mathcal{B}^l(\varepsilon(\mathbf{u}_*^l(t)), \alpha_*^l(t)), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + j(\dot{\mathbf{u}}_*(t), \mathbf{v})\\
&+ \sum_{l=1}^{2}\big((\mathcal{E}^l)^*\nabla\varphi_*^l(t), \varepsilon(\mathbf{v}^l)\big)_{\mathcal{H}^l} + j_d(\beta_*(t), \mathbf{u}_*(t), \mathbf{v}) = (\boldsymbol{f}(t), \mathbf{v})_V \\
&\qquad \forall\, \mathbf{v} \in V, \;\; \text{a.e. } t \in (0, T)\,.
\end{aligned} \tag{2.3.81}$$

Substituting (2.3.80) in (2.3.78) yields to

$$\begin{aligned}
&\alpha_*(t) \in \mathcal{K}, \quad \sum_{l=1}^{2}\big(\dot{\alpha}_*^l(t), \xi^l - \alpha_*^l(t)\big)_{L^2(\Omega^l)} + a(\alpha_*(t), \xi - \alpha_*(t)) \qquad (2.3.82)\\
&\geq \sum_{l=1}^{2}\big(\phi^l(\varepsilon(\mathbf{u_*}^1(t)), \alpha_*^l(t)), \xi^l - \alpha_*^l(t)\big)_{L^2(\Omega^l)} \quad \forall \xi \in \mathcal{K}, \; a.e. \; t \in (0, T)\,.
\end{aligned}$$

According to the relations (2.3.76)-(2.3.77) and (2.3.81)-(2.3.82) we can infer that $\{\mathbf{u}_*, \varphi_*, \beta_*, \alpha_*\}$ satisfies (2.3.26)-(2.3.29). Next, (2.3.30) and the regularities (2.3.31)-(2.3.34) follow from Lemmas 2.3.1, 2.3.2, 2.3.3, and 2.3.4. Since $\mathbf{u}_*$ and φ_* satisfy (2.3.31)-(2.3.32), it follows from (2.3.73) that $\boldsymbol{\sigma}_* \in C(0, T; \mathcal{H})$. Let now $t_1, t_2 \in [0, T]$, from (2.3.16), (3.2.78) and A(3)-A(4) we deduce that

$$|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)|_H \leq C\big(\,|\varphi_*(t_1) - \varphi_*(t_2)|_W + |\mathbf{u}_*(t_1) - \mathbf{u}_*(t_2)|_V\,\big). \tag{2.3.83}$$

The regularity of $\mathbf{u}_*$ and φ_* given by (2.3.31)-(2.3.32) and the inequality (2.3.83) entail $\boldsymbol{D}_* \in C(0, T; H)$. For $l = 1, 2$ we test (2.3.81) with $\mathbf{v}^l = \omega^l \in C_0^\infty(\Omega^l, \mathbb{R}^d)$, then we take $\psi^l \in C_0^\infty(\Omega^l)$ in (2.3.76) to obtain that

$$Div\,\boldsymbol{\sigma}_*^l(t) + \boldsymbol{f}_0^l(t) = 0, \quad div\,\boldsymbol{D}_*^l(t) = q_0^l(t) \quad a.e.\; t \in (0, T)\,.$$

Then, based on the assumptions in A(8), we conclude that $Div\,\boldsymbol{\sigma}_*^l \in C(0,T;H)$ and $div\,\boldsymbol{D}_*^l \in C(0,T : L^2(\Omega^l))$, and as a result $\boldsymbol{\sigma}_* \in C(0,T;\mathcal{H}_1)$ and $\boldsymbol{D}_* \in C(0,T;\mathcal{W})$. We deduce from the preceding that the weak solution $(\mathbf{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \boldsymbol{D}_*, \alpha_*, \beta_*)$ of the frictional contact problem P satisfies (2.3.31)-(2.3.36), which completes the existence part of the theorem. The uniqueness part results from of the uniqueness of the fixed point of the operator Λ defined by (2.3.65)-(2.3.67). ■

Chapter

3 Frictionless Contact Problem with Long Memory and Normal Compliance

Contents

3.1 Introduction

We look into the process of quasistatic frictionless contact between a viscoelastic piezoelectric body with long memory and a foundation, in this chapter. The contact is modeled with normal compliance. Here, the objective is to include material damage, numerically analyze the model, and interact mechanical and electrical fields with viscoelasticity under normal compliance condition. We offer a variational analysis of the mechanical problem and demonstrate that the model has a single weak solution. The problem is then numerically analyzed, and error estimates are derived for the numerical approximations based on discrete schemes.

This chapter is structured as follows. In Section 3.2 we present the variational analysis on electro-viscosity with long memory and normal compliance, we comment on the contact boundary conditions, determine the fundamental data assumptions, and develop the variational formulation before demonstrating the existence and uniqueness of the solution. In Section 3.3, we analyze a fully discrete scheme and derive error estimates under suitable solution regularity conditions.

3.2 Variational Analysis on Electro-Viscosity

Here, we concentrate on a frictionless contact problem between an electro-viscoelastic body and a deformable foundation with long memory and normal compliance condition. The problem is stated as a form of a coupled system for the displacement, stress, electric potential, and damage fields. We come at a variational formulation of the problem and demonstrate that the model has a single weak solution. This section is divided into three paragraphs. In the first paragraph, we propose the mechanical problem and indicate the hypotheses on the data. The variational formulation of the mechanical problem is then provided in the second paragraph. Finally, in the third paragraph, we demonstrate that the model has a single weak solution.

3.2.1 Mechanical Problem

The physical setting corresponds to that introduced in Subsection 1.2.1 As a reminder, We look at an electro-viscoelastic body that, in the reference configuration, resides in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary and dimensions Γ.

To provide the boundary condition for the problem, we divide Γ into three measurable and mutually disjoint parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$.

The unit outer normal on Γ is denoted by the symbol ν. Let $[0, T]$ represent the time interval of interest with $T > 0$. We suppose that the displacement disappears at $\Gamma_1 \times (0, T)$ where the body is clamped. A surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$, and volume force of density f_0 acts in $\Omega \times (0, T)$. We also suppose that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in contact with an insulator obstruction, the so-called foundation, on the potential contact surface Γ_3. The following is the classic formulation of the mechanical problem of electro-viscoelastic with damage.

Problem P

Determine a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$, a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$, an electric displacement field $\boldsymbol{D} : \Omega \times (0,T) \to \mathbb{R}^d$ an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, and a damage field $\alpha : \Omega \times (0,T) \to \mathbb{R}$,, such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\varepsilon\left(\boldsymbol{u}(t)\right) - \mathcal{E}^* E(\varphi) \\ + \int_0^t \mathcal{M}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \alpha(s)\right) ds\end{aligned} \quad \text{in } \Omega \times (0,T), \tag{3.2.1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}E(\varphi) \quad \text{in } \Omega \times (0,T), \tag{3.2.2}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_{\mathcal{K}}(\alpha) \ni \Theta\left(\varepsilon(\boldsymbol{u}), \alpha\right), \quad \text{in } \Omega \times (0,T), \tag{3.2.3}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.2.4}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{3.2.5}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.2.6}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.2.7}$$

$$-\sigma_\nu = p(u_\nu - g), \quad \boldsymbol{\sigma}_\tau = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{3.2.8}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0,T), \tag{3.2.9}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{3.2.10}$$

$$\boldsymbol{D}.\nu = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{3.2.11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \alpha(0) = \alpha_0, \quad \text{in } \Omega. \tag{3.2.12}$$

First, the electro-viscoelastic constitutive law with long term memory and damage is represented by equations (3.2.1) and (3.2.2), where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ are, respectively, nonlinear operators characterizing the purely viscous, the elastic, and the relaxation properties of the material, and α is the damage field. The electric field $E(\varphi) = -\nabla\varphi$, $\mathcal{E} = (g_{ijk})$ represent the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposition.

The evolution of the damage field is described in inclusion ((3.2.3), where $\varphi_{\mathcal{K}}(\alpha)$ indicates the subdifferential of the indicator function of the set $\mathcal{K}$ of acceptable damage functions established by

$$\mathcal{K} = \left\{\alpha \in H^1(\Omega) : 0 \leq \alpha \leq 1 \text{ a.e. in } \Omega\right\},$$

and Θ is the mechanical source of the damage.

Equations (3.2.4) and (3.2.5) represent the equilibrium equations for the stress and electric displacement fields. Equations (3.2.6)-(3.2.7) are the displacement-traction conditions.

The contact (3.2.8) is described in a normal compliance. Here, σ_ν stands for the normal stress, u_ν for the normal displacement, g for the gap between the body and the obstacle measured along the normal direction $\boldsymbol{\nu}$, and p is a predetermined function whose characteristics will be discussed below. Finally, we suppose that the contact is frictionless and therefore, $\boldsymbol{\sigma}_\tau = 0$.

A homogeneous Neumann boundary condition is described in relation (3.2.9). The electric boundary conditions are shown in (3.2.10) and (3.2.11). Finally, $\boldsymbol{u}_0$ is the initial displacement, and α_0 is the initial damage in equation (3.2.12).

3.2.2 Vartional Formulation

To follow the variational formulation of the problem (3.2.1)–(3.2.12), We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u}.\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{u}\| = (\boldsymbol{u}.\boldsymbol{u})^{1/2}, \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}.\boldsymbol{\tau})^{1/2}, \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Here and below the indices i, j and k run between 1 and d, unless otherwise stated, the summation convention over reiterated indices is used. Let $\Omega \subset \mathbb{S}^d (d = 1, 2, 3)$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We define the Hilbert spaces

$$H = L^2(\Omega)^d = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\}, \quad H_1 = \{\boldsymbol{u} = (u_i) \mid \varepsilon(\boldsymbol{u}) \in \mathcal{H}\},$$
$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\sigma} \in H\}.$$

and we are reminded of the fact that Green's formula holds

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{w}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{w})_H = \int_{\Gamma} \boldsymbol{\sigma}\nu.\boldsymbol{w} da, \quad \forall \boldsymbol{w} \in H_1.$$

where da is the surface measure element. Now, we introduce the closed subspace V of $H^1(\Omega)^d$ defined by

$$V = \left\{ \boldsymbol{w} \in H^1(\Omega)^d \mid \boldsymbol{w} = 0 \text{ on } \Gamma_1 \right\}.$$

We have meas $(\Gamma_1) > 0$, thus the following Korn's inequality holds,

$$\|\varepsilon(\boldsymbol{w})\|_{\mathcal{H}} \geq C_0 \|\boldsymbol{w}\|_{H^1(\Omega)^d}, \quad \forall \boldsymbol{w} \in V,$$

where c_0 stands for a positive constant that is dependent only on Ω and Γ_1

On V, we take into account the inner product and the accompanying norm provided by

$$(\boldsymbol{u}, \boldsymbol{w})_V = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{w}))_{\mathcal{H}}, \ \|\boldsymbol{w}\|_V = \|\varepsilon(\boldsymbol{w})\|_{\mathcal{H}}, \ \boldsymbol{u}, \boldsymbol{w} \in V. \tag{3.2.13}$$

Additionally, we present the spaces

$$W = \left\{ \zeta \in H^1(\Omega), \zeta = 0 \text{ on } \Gamma_a \right\},$$
$$\mathcal{W} = \left\{ \mathbf{D} \in H \mid \mathrm{div}\,\mathbf{D} \in L^2(\Omega) \right\},$$

where $\mathrm{div}\,\mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \xi)_W = \int_{\Omega} \nabla\varphi.\nabla\xi dx,$$
$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_{\Omega} \mathbf{D} \cdot \mathbf{E} dx + \int_{\Omega} \mathrm{div}\,\mathbf{D} \cdot \mathrm{div}\,\mathbf{E} dx.$$

The respective associated norms will be indicated by the notations $\|.\|_W$ and $\|.\|_{\mathcal{W}}$, respectively.

Since meas $(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\zeta\|_H \geq c_F\|\zeta\|_{H^1(\Omega)}, \quad \forall\zeta \in W, \tag{3.2.14}$$

where c_F stands for a positive constant that is dependent only on Ω and Γ_a.

Additionally, according to the Sobolev trace theorem, two positive constants, c_0 and $\tilde{c}_0$, exist such that

$$\|\boldsymbol{w}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{w}\|_V, \quad \forall\boldsymbol{w} \in V, \quad \|\phi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0\|\phi\|_W, \quad \forall\phi \in W. \tag{3.2.15}$$

In addition, if $\boldsymbol{D} \in \mathcal{W}$ is a regular function, so the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\xi)_H + (\operatorname{div}\boldsymbol{D}, \xi)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}\cdot\boldsymbol{\nu}\xi da, \quad \forall\xi \in H^1(\Omega). \tag{3.2.16}$$

When T > 0, we designate the space of continuous and continuously differentiable functions from $[0, T]$ to X, respectively,$C(0, T; X)$ and $C^1(0, T; X)$, with the norms

$$\|\boldsymbol{g}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{g}(t)\|_X.$$
$$\|\boldsymbol{g}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{g}(t)\|_X + \max_{t\in[0,T]} \|\dot{\boldsymbol{g}}(t)\|_X.$$

We take into account the following suppositions when studying the mechanical problem P

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There is } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \|\mathcal{B}(\boldsymbol{x}, \boldsymbol{\vartheta}_1) - \mathcal{B}(\boldsymbol{x}, \boldsymbol{\vartheta}_2)\| \leq L_{\mathcal{B}}\|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2\|, \\ \text{for all } \boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{\vartheta}) \text{ is Lebesgue measurable mapping on } \Omega, \\ \qquad \text{for all } \boldsymbol{\vartheta} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{3.2.17}$$

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \text{ There is } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\vartheta}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\vartheta}_2)\| \leq L_{\mathcal{A}} \|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2\|, \\ \quad \text{for all } \boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\vartheta}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\vartheta}_2)).(\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2) \geq m_{\mathcal{A}} \|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2\|^2, \\ \quad \text{for all } \boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (c) \;\; \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\vartheta}) \text{ is Lebesgue measurable mapping on } \Omega, \\ \quad \text{for any } \boldsymbol{\vartheta} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{3.2.18}$$

The relaxation function $\mathcal{M} : \Omega \times (0, T) \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There is a constant } L_{\mathcal{M}} > 0 \quad \text{such that} \\ \|\mathcal{M}(\boldsymbol{x}, t, \boldsymbol{\vartheta}_1, \alpha_1) - \mathcal{M}(\boldsymbol{x}, t, \boldsymbol{\vartheta}_2, \alpha_2)\| \leq L_{\mathcal{M}}(\|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2\| + \|\alpha_1 - \alpha_2\|), \\ \text{for all } t \in (0, T), \boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } \boldsymbol{x} \mapsto \mathcal{M}(\boldsymbol{x}, t, \boldsymbol{\vartheta}, \alpha) \text{ is Lebesgue measurable mapping on } \Omega, \\ \text{for all } \boldsymbol{\vartheta} \in \mathbb{S}^d, t \in (0, T), \text{ for all } \alpha \in \mathbb{R}. \\ \text{(c) } \boldsymbol{x} \mapsto \mathcal{M}(\boldsymbol{x}, t, \boldsymbol{\vartheta}, \alpha) \text{ is continuous mapping in } \Omega, \\ \text{for all } \boldsymbol{\vartheta} \in \mathbb{S}^d, t \in (0, T), \text{ for all } \alpha \in \mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{M}(\boldsymbol{x}, t, \mathbf{0}, 0) \in \mathcal{H}. \end{cases} \tag{3.2.19}$$

The piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{E} = (g_{ijk}), g_{ijk} \in L^\infty(\Omega), 1 \leq i, j, k \leq d. \\ (b) \quad \mathcal{E}(\mathrm{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{\tau}, \text{ for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{3.2.20}$$

Electric permittivity operator $\boldsymbol{B} = (p_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \boldsymbol{B}(\boldsymbol{\varepsilon}, E) = (p_{ij}(\boldsymbol{\varepsilon})E_j) \ \text{ for all } E = (E_i) \in \mathbb{R}^d, \ \text{ a.e. } \boldsymbol{\varepsilon} \in \Omega. \\ \text{(b) } p_{ij} = p_{ji} \in L^\infty(\Omega), 1 \le i, j \le d. \\ \text{(c) There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ \qquad \boldsymbol{B}E.E \ge m_{\boldsymbol{B}} \|E\|^2, \ \text{ for all } E = (E_i) \in \mathbb{R}^d, \ \text{ a.e. in } \Omega. \end{cases} \tag{3.2.21}$$

The function $\Theta : \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\Theta > 0 \quad \text{such that} \\ \|\Theta\left(\boldsymbol{x}, \boldsymbol{\vartheta}_1, \alpha_1\right) - \Theta\left(\boldsymbol{x}, \boldsymbol{\vartheta}_2, \alpha_2\right)\| \le L_\Theta\left(\|\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2\| + \|\alpha_1 - \alpha_2\|\right), \\ \text{for all } \boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \mathbb{S}^d, \ \text{ for all } \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } \boldsymbol{x} \mapsto \Theta(\boldsymbol{x}, \boldsymbol{\vartheta}, \alpha) \text{ is Lebesgue mapping measurable on } \Omega, \\ \qquad\qquad \text{for all } \boldsymbol{\vartheta} \in \mathbb{S}^d, \ \text{ for all } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \Theta(\boldsymbol{x}, \mathbf{0}, 0) \in L^2(\Omega). \end{cases} \tag{3.2.22}$$

The function $p_e : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $e = \nu, \tau$ satisfies

$$\begin{cases} \text{(a) There is } L_e > 0 \text{ such that} \\ \quad \|p_e\left(\boldsymbol{x}, r_1\right) - p_e\left(\boldsymbol{x}, r_2\right)\| \le L_e \left\|r_1 - r_2\right\| \\ \text{(b) } \left(p_e\left(\mathbf{x}, r_1\right) - p_e\left(\boldsymbol{x}, r_2\right)\right) \cdot \left(r_1 - r_2\right) \ge 0, \forall r_1, r_2 \in \mathbb{R}, \ \text{ a.e., on } \Gamma_3. \\ \text{for all } r_1, r_2 \in \mathbb{R}, \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(c) For any } r \in \mathbb{R}, \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \text{(d) } \boldsymbol{x} \mapsto p_e(\boldsymbol{x}, 0) \text{is a map belongs to } L^2(\Gamma_3). \end{cases} \tag{3.2.23}$$

We suppose that the initial data $\boldsymbol{u}_0$ and α_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2, satisfy

$$\boldsymbol{u}_0 \in V, \quad \alpha_0 \in K, \tag{3.2.24}$$

$$f_0 \in C\left(0,T; L^2(\Omega)^d\right), \ f_2 \in C\left(0,T; L^2\left(\Gamma_2\right)^d\right), \tag{3.2.25}$$

$$q_0 \in C\left(0,T; L^2(\Omega)\right), \ q_2 \in C\left(0,T; L^2\left(\Gamma_b\right)\right). \tag{3.2.26}$$

We have the bilinear form $a : E \times E \to \mathbb{R}$ given by

$$a(\phi, \psi) = k \int_\Omega \nabla\phi.\nabla\psi dx, \tag{3.2.27}$$

where, the microcrack diffusion coefficient verifies: $k > 0$

Next. three mappings $j : V \times V \to \mathbb{R},\ f : [0,T] \to V$, $q : [0,T] \to W$, are defined by

$$j(\boldsymbol{u}, \boldsymbol{w}) = \int_{\Gamma_3} p\left(u_\nu - g\right) w_\nu \,\mathrm{d}a, \tag{3.2.28}$$

$$(\boldsymbol{f}(t), \boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{w} \,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{w} \,\mathrm{d}a, \tag{3.2.29}$$

$$(q(t), \xi)_W = \int_\Omega q_0(t)\xi \,\mathrm{d}x - \int_{\Gamma_b} q_2(t)\xi \,\mathrm{d}a, \tag{3.2.30}$$

for all $\boldsymbol{u}, \boldsymbol{w} \in V$, $\xi \in W$ and $t \in [0,T]$. Note that

$$\boldsymbol{f} \in C\left(0,T;V\right), \quad q \in C(0,T;W). \tag{3.2.31}$$

Standard arguments allow us to come at the variational formulation of the quasistatic frictionless problem with long memory, normal compliance, and damage, which is as follows

Problem PV

Determine a stress field $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, a displacement field $\boldsymbol{u} : (0,T) \to V$, an electric potential $\varphi : (0,T) \to W$, and a damage field $\alpha : (0,T) \to H^1(\Omega)$, such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\varepsilon\left(\boldsymbol{u}(t)\right) + \mathcal{E}^*\nabla\varphi(t) \\ + \int_0^t \mathcal{M}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \alpha(s)\right) ds,\end{aligned} \tag{3.2.32}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{w}))_{\mathcal{H}} + j(u(t), \boldsymbol{w}) = (\mathbf{f}(t), \boldsymbol{w})_V, \quad \forall \boldsymbol{w} \in V, \tag{3.2.33}$$

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall\phi \in W, t \in (0,T), \tag{3.2.34}$$

$$\begin{aligned}&\alpha(t) \in \mathcal{K}, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\geq \left(\Theta\left(\varepsilon(\boldsymbol{u}(t)), \alpha(t)\right), \xi - \alpha(t)\right)_{L^2(\Omega)}, \forall \xi \in \mathcal{K}, t \in (0,T),\end{aligned} \tag{3.2.35}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \alpha(0) = \alpha_0. \tag{3.2.36}$$

The next subsection contains our main existence and uniqueness result for Problem P V.

3.2.3 Existence and Uniqueness Results

Theorem 3.2.1 *suppose (3.2.18)-(3.2.26) are true. Then the problem PV has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \boldsymbol{D})$. Additionally, the solution has the regularity*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{3.2.37}$$

$$\varphi \in C(0,T;W), \tag{3.2.38}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}), \tag{3.2.39}$$

$$\alpha \in H^1\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{3.2.40}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{3.2.41}$$

The solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \boldsymbol{D})$ that satisfies (3.2.32)-(3.2.36) is said to as a weak solution to the contact problem P. We come to conclusion that the mechanical problem (3.2.1)-(3.2.12) has a unique weak solution satisfying (3.2.37)-(3.2.41), under the mentioned assumptions.

Theorem 3.2.1 will be proved in steps using arguments from evolution equations with monotone operators, a standard existence and uniqueness result on parabolic inequalities, and fixed-point.

In this section, C will stand for a strictly positive integer whose value may vary depending on the problem's data but is independent of time.

Let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ be provided. The following variational problem is considered in the first step

Problem $\mathcal{P}_\eta^1$

Determine a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ such that for all $t \in [0,T]$

$$\begin{aligned} & \left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_\eta(t)\right), \varepsilon\left(\boldsymbol{w}\right)\right)_{\mathcal{H}} + \left(\boldsymbol{\eta}(t), \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}} = \left(\boldsymbol{f}(t), \boldsymbol{w}\right)_V, \\ & \forall \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T), \end{aligned} \tag{3.2.42}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{3.2.43}$$

The study of Problem $\mathcal{P}_\eta^1$ produced the next result

Lemma 3.2.1 *Problem $\mathcal{P}_\eta^1$ has an unique solution, $\boldsymbol{u}_\eta \in C^1(0,T;V)$*

Proof. We use Riesz Representation Theorem to determine the element $f_\eta \in C(0,T;V)$ and the operator $A : V \to V$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{3.2.44}$$

$$(\mathrm{f}_\eta(t), \boldsymbol{w})_V = (\mathbf{f}(t), \boldsymbol{w})_V - (\boldsymbol{\eta}(t), \boldsymbol{w})_V \quad \text{a.e.t } \in [0,T]. \tag{3.2.45}$$

For all $\mathbf{w} \in V$ we have

$$\begin{aligned}\|(A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{w})_V\| &= \|(\mathcal{A}\varepsilon(\boldsymbol{u})-\mathcal{A}\varepsilon(\boldsymbol{v}),\varepsilon(\boldsymbol{w}))_{\mathcal{H}}\| \\ &\leq \|\mathcal{A}\varepsilon(\boldsymbol{u})-\mathcal{A}\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}\|\varepsilon(\boldsymbol{w})\|_{\mathcal{H}} \\ &\leq L_{\mathcal{A}}\|\boldsymbol{u}-\boldsymbol{v}\|_V\|\boldsymbol{w}\|_{\mathcal{H}}, \quad \forall \boldsymbol{u},\boldsymbol{v}\in V,\end{aligned}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by (3.2.45) and (3.2.18), we find

$$\begin{aligned}(A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})_V &= \left(\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}\right)\right)-\mathcal{A}\left(\varepsilon\left(\boldsymbol{v}\right)\right),\varepsilon\left(\boldsymbol{u}\right)-\varepsilon\left(\boldsymbol{v}\right)\right)_{\mathcal{H}} \\ &\geq m_{\mathcal{A}}\left\|\varepsilon\left(\boldsymbol{u}\right)-\varepsilon\left(\boldsymbol{v}\right)\right\|_{\mathcal{H}}^2 \geq C\left\|\boldsymbol{u}-\boldsymbol{v}\right\|_V^2,\end{aligned}$$

i.e., that $A : V \to V$is a strongly monotone operator on V. As a result, A is inversible and its inverse A^{-1} is also strongly monotone Lipschitz continuous on V.

Therefore, there exists a unique function $\boldsymbol{v}_\eta \in C(0,T;V)$ that satisfies

$$A\boldsymbol{v}_\eta(t) = \mathbf{f}_\eta(t). \tag{3.2.46}$$

We consider the function $\boldsymbol{u}_\eta : [0,T] \to V$ defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0, \forall t \in [0,T]. \tag{3.2.47}$$

It follows from (3.2.42)-(3.2.47) that $\boldsymbol{u}_\eta$ is a solution of P_η^1 and it satisfies (3.2.37), then the existence part of lemma 3.2.1 is now complete. The uniqueness of the solution follows from the uniqueness of the solution of the problem (3.2.46). ■

We use the solution $\boldsymbol{u}_\eta$, obtained in Lemma 3.2.1, to construct the following variational problem for the electrical potential in the second step.

Problem $\mathcal{P}_\eta^2$

Determine an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\xi)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\xi)_H = (q(t), \xi)_W, \text{ for all } \xi \in W, t \in (0,T). \tag{3.2.48}$$

We have the following existence and uniqueness result for problem $\mathcal{P}_\eta^2$

Lemma 3.2.2 *Problem (3.2.48) has a unique solution φ_η that satisfies the regularity (3.2.38). Additionally, if φ_{η_i} represents the solutions of problem (3.2.48) for $\eta_i \in C([0,T];\mathcal{H}), i = 1,2,$ then there exists $C > 0$ such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq C \|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V, \quad \forall t \in [0,T]. \tag{3.2.49}$$

Proof. Let $\Psi : W \times W \to \mathbb{R}$ be the form given by

$$\Psi(\varphi, \xi) = (\mathbf{B}\nabla\varphi, \nabla\xi)_H, \quad \forall \varphi, \xi \in W, \tag{3.2.50}$$

We show that the form Ψ is symmetric, coercive and bilinear continuous on W, using (3.2.14), (3.2.21), (3.2.50) and defined $(\varphi, \xi)_W$ In addition, we may define an element $\Phi_\eta : [0,T] \to W$ using (3.2.29) and the Riesz Representation Theorem, in order that

$$(\Phi_\eta(t), \xi)_W = (q(t), \xi)_W + (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\xi)_H, \quad \forall \xi \in W, t \in (0,T),$$

By applying the Lax-Milgram Theorem, we may conclude there is only one element $\varphi_\eta(t) \in W$ such that

$$\Psi(\varphi_\eta(t), \xi) = (\Phi_\eta(t), \xi)_W, \quad \forall \xi \in W. \tag{3.2.51}$$

It follows from (3.2.51) that φ_η is a solution of the equation (3.2.48). Let $\varphi_{\eta_i} = \varphi_i$, and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1, 2$. We use (3.2.48) to obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leq C \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V , \quad \forall t \in [0, T].$$

Considering that $\boldsymbol{u}_\eta \in C^1(0, T; V)$ implies that $\varphi_\eta \in C(0, T; W)$, and then the proof is now complete. ■

We let $\theta \in C(0, T; L^2(\Omega))$, and construct the following problem for the damage field, in the third step

Problem $\mathcal{P}_\theta$

Determine the damage field $\alpha_\theta : (0, T) \to L^2(\Omega)$ such that $\alpha_\theta(t) \in \mathcal{K}$ and

$$\begin{aligned} &(\dot{\alpha}_\theta(t), \xi - \alpha_\theta)_{L^2(\Omega)} + a\left(\alpha_\theta(t), \xi - \alpha_\theta(t)\right) \\ &\qquad \geq (\theta(t), \xi - \alpha_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in \mathcal{K}, \text{ a.e. } t \in (0, T), \end{aligned} \tag{3.2.52}$$

$$\alpha_\theta(0) = \alpha_0. \tag{3.2.53}$$

We apply Theorem 1.3.7 to problem $\mathcal{P}_\theta$

Lemma 3.2.3 *The auxiliary problem $\mathcal{P}_\theta$ has an unique solution α_θ and it satisfies (3.2.40).*

Proof. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, the inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. Then we can write the Gelfand triple

$$E \subset Y \subset E'.$$

We denote by $(.,.)_{(H^1(\Omega))'\times H^1(\Omega)}$ the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$, we note that $\mathcal{K}$ is a closed convex set in$(H^1(\Omega))$, thus we have

$$(\alpha,\zeta)_{(H^1(\Omega))'\times H^1(\Omega)} = (\alpha,\zeta)_{L^2(\Omega)}, \quad \forall\alpha\in L^2(\Omega), \zeta\in H^1(\Omega),$$

Taking into consideration the fact that $\alpha_\theta \in \mathcal{K}$ in (3.2.24), using (3.2.27) the definition of the bilinear form a, leads us to see that Lemma 2.3.4 is a direct consequence of Theorem 1.3.7. ■

Finally, in light of these results and utilizing the characteristics of the operators $\mathcal{M}$, $\mathcal{E}$ as well as the function Θ for $t\in(0,T)$, we consider the operator

$$\Lambda : C\left(0,T;\mathcal{H}\times L^2(\Omega)\right) \to C\left(0,T;\mathcal{H}\times L^2(\Omega)\right) \tag{3.2.54}$$

defined by

$$\Lambda(\eta,\theta)(t) = \left(\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)\right) \in \mathcal{H}\times L^2(\Omega) \tag{3.2.55}$$

and

$$\begin{aligned}\left(\Lambda^1(\eta,\theta)(t), v\right)_{\mathcal{H}\times V} &= \left(\mathcal{B}\left(\varepsilon\left(\boldsymbol{u}_\eta(t)\right)\right), \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}} + \left(\mathcal{E}^*\nabla\varphi_\eta(t), \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}} \\ &+ \left(\int_0^t \mathcal{M}(t-s), \varepsilon\left(\boldsymbol{u}_\eta(s), \alpha(s)\right) ds, \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}} + j(\boldsymbol{u}_\eta(t), \boldsymbol{w}), \quad \forall \boldsymbol{w}\in V\end{aligned} \tag{3.2.56}$$

$$\Lambda^2(\eta,\theta)(t) = \Theta\left(\varepsilon\left(\boldsymbol{u}_\eta(t)\right), \alpha_\theta(t)\right). \tag{3.2.57}$$

Here, for each $(\eta,\theta)\in C(0,T;\mathcal{H}\times L^2(\Omega))$, the displacement field, the electric potential field and the damage field determined in Lemmas 3.2.1, 3.2.2 and 3.2.3 respectively, are represented by the symbols $\boldsymbol{u}_\eta$, φ_η, and α_θ represent We have the following result.

Lemma 3.2.4 *The operator* Λ *has a fixed point* $(\boldsymbol{\eta}^*,\theta^*)\in C(0,T;\mathcal{H}\times L^2(\Omega))$, *such that* $\Lambda(\boldsymbol{\eta}^*,\theta^*) = (\boldsymbol{\eta}^*,\theta^*)$.

Proof. Let $({\eta^*}_1,\theta_1^*), ({\eta^*}_2,\theta_2^*) \in C\left(0,T;\mathcal{H}\times L^2(\Omega)\right)$ and $t\in(0,T)$. For $i=1,2$, we use the notation: $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \dot{\boldsymbol{u}}_i = \boldsymbol{v}_i$, $\alpha_{\theta_i} = \alpha_i$, and $\varphi_{\eta_i} = \varphi_i$, Let us start by using (3.2.15), (3.2.17), (3.2.19), (3.2.20) and (3.2.23), we have

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 \leq \|\mathcal{B}\varepsilon(\boldsymbol{u}_1(t))-\mathcal{B}\varepsilon(\boldsymbol{u}_2(t))\|_{\mathcal{H}}^2\\ &\quad+\int_0^t \|\mathcal{M}\left(t-s,\varepsilon(\boldsymbol{u}_1(s),\alpha_1(s))\right)-\mathcal{M}\left(t-s,\varepsilon(\boldsymbol{u}_2(s)),\alpha_2(s)\right)\|_{\mathcal{H}}^2\,ds\\ &\quad+\|\mathcal{E}^*\nabla\varphi_1(t)-\mathcal{E}^*\nabla\varphi_2(t)\|_{\mathcal{H}}^2+\|p(u_{1\nu}-g)-p(u_{2\nu}-g)\|_{\mathcal{H}}^2\\ &\quad\leq C\left(\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\mathcal{H}}^2\right.\\ &\quad\left.+\int_0^t \|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\right).\end{aligned} \tag{3.2.58}$$

We use estimate (3.2.49), to obtain

$$\begin{aligned}\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_{\mathcal{H}}^2 &\leq C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\mathcal{H}}^2\right.\\ &\left.+\int_0^t \|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\right).\end{aligned} \tag{3.2.59}$$

By similar arguments, from (3.2.57), (3.2.22) we obtain

$$\begin{aligned}&\|\Lambda^2(\boldsymbol{\eta}_1,\lambda_1)(t)-\Lambda^2(\boldsymbol{\eta}_2,\lambda_2)(t)\|_{\mathcal{H}}^2\\ &\quad\leq C(|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2),\quad \text{a.e. } t\in(0,T).\end{aligned} \tag{3.2.60}$$

It follows now from (3.2.55), (3.2.59) and (3.2.60) that

$$\begin{aligned}&\|\Lambda\left(\boldsymbol{\eta}_1,\lambda_1\right)(t)-\Lambda\left(\boldsymbol{\eta}_2,\lambda_2\right)(t)\|_{\mathcal{H}\times L^2(\Omega)}^2\\ &\qquad\leq C(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2\\ &\qquad+\int_0^t \|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2+\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)ds\|_V^2)\end{aligned} \tag{3.2.61}$$

Since $\boldsymbol{u}_\eta(t)=\int_0^t \boldsymbol{v}_\eta(s)ds+\boldsymbol{u}_0\,\forall t\in[0,T]$, we obtain

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2 \leq \int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2\,ds,\quad \forall t\in[0,T]. \tag{3.2.62}$$

Moreover, from (3.2.42) we obtain that

$$(\mathcal{A}\varepsilon(\boldsymbol{v}_1) - \mathcal{A}\varepsilon(\boldsymbol{v}_2), \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} + (\eta_1 - \eta_2, \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} = 0.$$

Using the assumption (3.2.18), we determine that

$$\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_V^2 \leq C\,\|\eta_1(t) - \eta_2(t)\|_{\mathcal{H}}^2 . \qquad (3.2.63)$$

We conclude from (3.2.52) that

$$\begin{aligned}(\theta_1 - \theta_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} &\geq (\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} \\ &+ a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2), \text{ a.e. } t \in (0,T).\end{aligned}$$

Using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$, integrating this inequality with respect to time, keeping in mind $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \geq 0$ leads us to discover that

$$\tfrac{1}{2}\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \leq C\int_0^t (\theta_1(s) - \theta_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds.$$

This implies that

$$\begin{aligned}&\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \\ &\qquad \leq C\left(\int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Omega)}^2 ds\right).\end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)}^2 \leq C\int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds \;, \forall t \in [0,T]. \qquad (3.2.64)$$

Form estimates (3.2.61), (3.2.63) and the previous inequality it follows now that

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)}\\&\qquad\le C\int_0^T\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)}\,ds.\end{aligned}\tag{3.2.65}$$

If we repeat this inequality m times, we get

$$\begin{aligned}&\|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega))}\\&\qquad\le\frac{C^mT^m}{m!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H}\times L^2(\Omega))$, and so Λ has a unique fixed point. ■

We now have all the necessary elements to demonstrate Theorem 3.2.1.

Existence

Let $(\eta^*,\theta^*)\in H^2(0,T;\mathcal{H}\times L^2(\Omega))$, be the fixed point of Λ and denote

$$\boldsymbol{u}=\boldsymbol{u}_{\eta^*},\quad \alpha=\alpha_{\theta^*},\quad \varphi=\varphi_{\eta^*},\tag{3.2.66}$$

$$\boldsymbol{\sigma}=\mathcal{A}\varepsilon(\dot{\boldsymbol{u}})+\mathcal{B}\varepsilon(\boldsymbol{u})+\mathcal{E}^*\nabla\varphi(t)+\int_0^t\mathcal{M}(t-s,\varepsilon(\boldsymbol{u}(s)),\alpha(s))\,ds,\tag{3.2.67}$$

$$\boldsymbol{D}=\mathcal{E}\varepsilon(\boldsymbol{u})+\boldsymbol{B}\nabla(\varphi)\tag{3.2.68}$$

we use : $\Lambda^1(\eta^*,\theta^*)=\eta^*$, $\Lambda^2(\eta^*,\theta^*)=\theta^*$, it follows

$$\begin{aligned}(\eta^*(t),\boldsymbol{w})_V&=(\mathcal{B}(\varepsilon(\boldsymbol{u}(t))),\varepsilon(\boldsymbol{w}))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi(t),\varepsilon(\boldsymbol{w}))_{\mathcal{H}}\\&\quad+j(\boldsymbol{u}(t),\boldsymbol{w})+\left(\int_0^t\mathcal{M}(t-s),\varepsilon(\boldsymbol{u}(s),\alpha(s))\,ds,\varepsilon(\boldsymbol{w})\right)_{\mathcal{H}},\forall\boldsymbol{w}\in V.\end{aligned}\tag{3.2.69}$$

$$\theta^*(t)=\Theta(\boldsymbol{u}(t),\alpha(t)).\tag{3.2.70}$$

We demonstrate that (u,,,, D) meets the regularities (3.2.32) to (3.2.36) and (3.2.37). (3.2.41). In fact, we write (3.2.42) for = and apply (3.2.63) to find

We demonstrate that $(\boldsymbol{u}, \boldsymbol{\sigma}, \alpha, \varphi, \boldsymbol{D})$ meets (3.2.33)–(3.2.36) and the regularities (3.2.37)–(3.2.41). In fact, we write (3.2.42) for $\eta = \eta^*$ and apply (3.2.63) to obtain

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{w}))_{\mathcal{H}} + (\boldsymbol{\eta}^*(t), \boldsymbol{w})_{\mathcal{H}} = (\boldsymbol{f}(t), \boldsymbol{w})_V, \\ \text{a.e. } t \in (0,T), \text{ for all } \boldsymbol{w} \in V.\end{gathered} \tag{3.2.71}$$

Substitute (3.2.69) in (3.2.71) to obtain

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{w}))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\boldsymbol{u}(t)), \varepsilon(\boldsymbol{w}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{w}))_{\mathcal{H}} \\ + \left(\int_0^t \mathcal{F}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \alpha(s)\right) ds, \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}} + j\left(\boldsymbol{u}(t), \boldsymbol{w}\right) \\ = (\boldsymbol{f}(t), \boldsymbol{w})_V, \quad \text{a.e. } t \in (0,T), \text{ for all } \boldsymbol{v} \in V.\end{gathered} \tag{3.2.72}$$

and we write (3.2.52) for $\theta = \theta^*$ and use (3.2.66) to find

$$\begin{gathered}\alpha(t) \in \mathcal{K}, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a\left(\alpha(t), \xi - \alpha(t)\right) \\ \geq (\theta^*(t), \xi - \alpha(t))_{L^2(\Omega)}, \forall \xi \in \mathcal{K}, \text{ a.e.t } \in (0,T).\end{gathered} \tag{3.2.73}$$

We substitute (3.2.70) in (3.2.73) to obtain

$$\begin{gathered}\alpha(t) \in \mathcal{K}, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a\left(\alpha(t), \xi - \alpha(t)\right) \\ \geq \left(\Theta\left(\boldsymbol{u}\left(t\right), \alpha\left(t\right)\right), \xi - \alpha(t)\right)_{L^2(\Omega)}, \forall \xi \in \mathcal{K}, \text{ a.e.t } \in (0,T).\end{gathered} \tag{3.2.74}$$

We write now (3.2.48) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (3.2.66) to find

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}(t)), \nabla\phi)_H = (q(t), \phi)_W, \quad \forall \phi \in W, t \in (0,T) \tag{3.2.75}$$

Given the relations (3.2.71)-(3.2.75), we can now say that $(\boldsymbol{u}, \boldsymbol{\sigma}, \alpha, \varphi, \boldsymbol{D})$ satisfies (3.2.32)-(3.2.35). Since $\boldsymbol{u}, \varphi$ and α satisfies (3.2.37)-(3.2.38) and (3.2.40) , respectively, we can conclude from (3.2.67) that

$$\boldsymbol{\sigma} \in C(0,T;H), \tag{3.2.76}$$

we select $\boldsymbol{v} = \boldsymbol{u} \pm \phi$ in (2.3.68), with $\phi \in D(\Omega)^d$ to obtain

$$\operatorname{Div} \boldsymbol{\sigma}(t) = f_0(t), \forall t \in [0,T], \tag{3.2.77}$$

where $D(\Omega)$ represents the space of infinitely differentiable real functions with a compact support in Ω. The regularity (3.2.39) follows from (3.2.25), (3.2.76) and (3.2.77).

Let now $t_1, t_2 \in [0,T]$, from (3.2.14), (3.2.21), (3.2.20) and (3.2.68), we infer that there exists a positive constant $C > 0$ that satisfies

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \leq C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V\right).$$

The regularity of $\boldsymbol{u}$ and φ implied by (3.2.37) and (3.2.38) implies

$$\boldsymbol{D} \in C(0,T;H). \tag{3.2.78}$$

Using (3.2.30) and choosing $\phi \in D(\Omega)^d$ lead us to obtain

$$\operatorname{div} \boldsymbol{D}(t) = q_0(t), \quad \forall t \in [0,T], \tag{3.2.79}$$

The existence part of the theorem 3.2.1 is concluded by the property $\boldsymbol{D} \in C(0,T;\mathcal{W})$, which follows from (3.2.26),(3.2.78) and (3.2.79).

Uniqueness

The proof is complete since the uniqueness of the solution follows from the uniqueness of the fixed point of operator Λ.

3.3 Numerical Analysis on Electro-Viscosity

In this section, we introduce a fully scheme for the numerical solutions of the frictionless contact problem between an electro-viscoelastic body with long memory and a deformable foundation. This scheme is based on the Euler scheme to discretize the time derivatives and the finite element method to approximate the spatial variable. Next, we get error estimates for the approximate solutions. The simulation of certain numerical results for two-dimensional test problems is offered at the end.

This section is divided into three paragraphs. In the first paragraph, we introduce finite element spaces and basic notations. In the second paragraph we give the approximate variational formulation of the mechanical problem. Finally, in the third paragraph, we present error estimates on the approximate solutions.

3.3.1 Notations and Discrete Spaces

We employ uniform partitions of the time interval $[0, T] : 0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n = nk$, and $k = T/N$ is the time step characterizing the partition. We write $f_n = f(t_n)$ for a continuous function $f(t)$. Additionally, letlet $\delta w_n = (w_n - w_{n-1})/k$ signify the divided differences for a sequence $\{w_n\}_{n=0}^{N}$.

The tangential part of $\boldsymbol{u}_n$ on the boundary and its normal component are indicated by

$$u_{n,\nu} = (\boldsymbol{u}_n)_\nu \equiv \boldsymbol{u}_n \cdot \boldsymbol{\nu},$$

$$\boldsymbol{u}_{n,\tau} = (\boldsymbol{u}_n)_\tau \equiv \boldsymbol{u}_n - u_{n,\nu}\boldsymbol{\nu},$$

Fully discrete approximations will be associated with the superscript "hk" to indicate the discretizations in space (h) and in time (k). As an example, $\boldsymbol{u}_n^{hk}$ is an approximation of $\boldsymbol{u}_n$, and its normal component and tangential part on the boundary are denoted by $u_{n,\nu}^{hk}$ and $\boldsymbol{u}_{n,\tau}^{hk}$. Note that this notation will be used only when we deal with the numerical analysis of problems.

We will suppose that the domain Ω is polygonal or polyhedral its boundary Γ is divided into three subsets Γ_1, Γ_2 and Γ_3, with mutually disjoint interiors, when we discretize the spatial derivatives. The unit outward normal vector is constant for each Γ_j^i, thus we write $\Gamma_j = \bigcup_{i=1}^{i_j} \Gamma_j^i$. Let $\mathcal{T}^h$ be a regular family of finite element partitions of $\overline{\Omega}$ that are presumed to be consistent with the decomposition $\Gamma = \bigcup_{j=1}^{3} \Gamma_j$, in the sense that if the intersection of one side of an element with one of the three sets has a positive surface measure, then the side lies wholly in that set. We have introduced above in Section 3.2 the following spaces and we recall them here

$$
\begin{aligned}
V &= \left\{ \boldsymbol{w} \in H^1(\Omega)^d : \boldsymbol{w} = 0 \text{ on } \Gamma_1 \right\}, \\
W &= \left\{ \zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \Gamma_a \right\}, \\
\mathcal{K} &= \left\{ \alpha \in H^1(\Omega) : 0 \leq \alpha \leq 1 \text{ a.e. in } \Omega \right\}.
\end{aligned}
$$

Let $V^h \subset V$, $W^h \subset W$ and $\mathcal{K}^h \subset \mathcal{K}$ be a finite-dimensional spaces defined by

$$
\begin{aligned}
V^h &= \left\{ \boldsymbol{w}^h \in C(\overline{\Omega})^d : \boldsymbol{\omega}^h_{|K} \text{ linear } \forall K \in \mathcal{T}^h,\ \boldsymbol{w}^h = 0 \text{ on } \Gamma_1 \right\}, \\
W^h &= \left\{ \zeta^h \in C(\overline{\Omega}) : \zeta^h_{|K} \text{ linear } \forall K \in \mathcal{T}^h,\ \zeta^h = 0 \text{ on } \Gamma_a \right\}, \\
\mathcal{K}^h &= \left\{ \alpha^h \in C(\overline{\Omega}) : \alpha^h_{|K} \text{ linear } \forall K \in \mathcal{T}^h \right\}.
\end{aligned}
$$

Then , using the above finite element discretization for the spatial variables and the backward Euler scheme for the time derivatives, we obtain the following fully discrete scheme which approximates Problem PV.

3.3.2 Discrete Problem

Problem PV^{hk}

Determine a discrete displacement field $\boldsymbol{u}^{hk} = \left\{ \boldsymbol{u}_n^{hk} \right\}_{n=0}^{N} \subset V^h$, a discrete damage field $\alpha^{hk} = \left\{ \alpha_n^{hk} \right\}_{n=0}^{N} \subset \mathcal{K}^h$ and a discrete electric potential field $\varphi^{hk} = \left\{ \varphi_n^{hk} \right\}_{n=0}^{N} \subset W^h$, such that

$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h$, $\alpha_0^{hk} = \alpha_0^h$ and for $n = 1, ..., N$

$$\begin{aligned}&\left(\delta\alpha_n^{hk}, \xi^h - \alpha_n^{hk}\right)_{L^2(\Omega)} + a\left(\alpha_n^{hk}, \xi^h - \alpha_n^{hk}\right)\\ &\qquad\geq \left(\Theta\left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{n-1}^{hk}\right), \alpha_{n-1}^{hk}\right), \xi^h - \alpha_n^{hk}\right)_{L^2(\Omega)}, \qquad \forall\xi^h \in \mathcal{K}^h,\end{aligned} \tag{3.3.1}$$

$$\begin{aligned}&\left(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}} + \left(\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{n-1}^{hk}\right), \boldsymbol{\varepsilon}\left(w^h\right)\right)_{\mathcal{H}} + \left(\mathcal{E}^*\nabla\varphi_n^{hk}, \varepsilon\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}} + j\left(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h\right)\\ &= \left(f_n, \boldsymbol{w}^h\right)_V - \left(\sum_{j=1}^{n} k\mathcal{M}\left(t_n - t_{j-1}, \boldsymbol{\varepsilon}\left(u_{j-1}^{hk}\right), \alpha_{j-1}^{hk}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}} \qquad \forall\boldsymbol{w}^h \in V^h,\end{aligned} \tag{3.3.2}$$

$$\left(\boldsymbol{\beta}\nabla\varphi_n^{hk}, \nabla\psi^h\right)_H - \left(\mathcal{E}\varepsilon\left(\boldsymbol{u}_n^{hk}\right), \nabla\psi^h\right)_H = \left(q_n, \psi^h\right)_W \quad \forall\psi^h \in W^h. \tag{3.3.3}$$

We discover that Problem PV^{hk} admits a single solution using classic results of nonlinear variational equations (see [32]), which we summarize below.

Theorem 3.3.1 *Assume that (3.2.18)–(3.2.23) hold. Then, Problem PV^{hk} has a unique solution $(\boldsymbol{u}^{hk}, \varphi^{hk}, \alpha^{hk}) \subset V^h \times W^h \times \mathcal{K}^h$.*

3.3.3 Numerical Approximation

The purpose of this section is to to estimate the numerical errors $\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V$, $\|\varphi_n - \varphi_n^{hk}\|_V$ and $\|\alpha_n - \alpha_n^{hk}\|_Y$. Let's start by getting an error estimation on the electric potential. As we continue as in [7], by subtracting (3.3.3) from (3.2.34) at time $t = t_n$ for $\psi = \psi^h \in W^h$, it is evident that

$$\left(\boldsymbol{\beta}\nabla\left(\varphi_n - \varphi_n^{hk}\right), \nabla\psi^h\right)_H - \left(\mathcal{E}\left(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right), \nabla\psi^h\right)_H = 0, \quad \forall\psi^h \in W^h.$$

Thus, we have

$$\begin{aligned}&\left(\boldsymbol{\beta}\nabla\left(\varphi_n - \varphi_n^{hk}\right), \nabla\left(\varphi_n - \varphi_n^{hk}\right)\right)_H - \left(\mathcal{E}\left(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right), \nabla\left(\varphi_n - \varphi_n^{hk}\right)\right)_H\\ &= \left(\boldsymbol{\beta}\nabla\left(\varphi_n - \varphi_n^{hk}\right), \nabla\left(\varphi_n - \psi^h\right)\right)_H - \left(\mathcal{E}\left(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right), \nabla\left(\varphi_n - \psi^h\right)\right)_H, \forall\psi^h \in W^h.\end{aligned}$$

We apply the Cauchy's inequality and use properties (3.2.21) and (3.2.20), after some algebra to find that

$$\left\|\varphi_n-\varphi_n^{hk}\right\|_V^2 \leq c\left(\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+\left\|\varphi_n-\psi^h\right\|_W^2\right), \quad \forall \psi^h \in W^h. \tag{3.3.4}$$

Secondly, allow us to estimate the numerical errors on the displacement field. In order to have, we therefore replace (3.2.32) in (3.2.33). (3.2.32) in (3.2.33)

$$\begin{aligned} & (\mathcal{A} \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}+(\mathcal{B} \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}+\left(\mathcal{E}^* \nabla \varphi, \varepsilon(\boldsymbol{w})\right)_{\mathcal{H}}+j(\boldsymbol{u}(t), \boldsymbol{w}) \\ & \quad+\left(\int_0^t \mathcal{M}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \alpha(s)) d s, \boldsymbol{w}\right)_{\mathcal{H}}=(\mathbf{f}(t), \boldsymbol{w})_V, \quad \forall \boldsymbol{w} \in V, \end{aligned} \tag{3.3.5}$$

We rewrite (3.3.5) at time $t=t_n$ for $\boldsymbol{w}=\boldsymbol{w}^h \in V^h$ and we subtract it to variational equation (3.3.2) to find

$$\begin{aligned} & \left(\mathcal{A} \varepsilon\left(\dot{\boldsymbol{u}}_n-\delta \boldsymbol{u}_n^{hk}\right), \varepsilon\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}}+\left(\mathcal{B} \varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right), \varepsilon\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}} \\ & +\left(\mathcal{E}^* \nabla(\varphi_n-\varphi_n^{hk}), \varepsilon\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}}+j\left(\boldsymbol{u}_n, \boldsymbol{w}^h\right) \\ & -j\left(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h\right)+\left(\int_0^{t_n} \mathcal{M}(t_n-s, \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \alpha(s)) d s\right. \\ & \left.-\sum_{j=1}^n k \mathcal{M}\left(t_n-t_{j-1}, \boldsymbol{\varepsilon}\left(\boldsymbol{u}_{j-1}^{hk}\right), \alpha_{j-1}^{hk}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{w}^h\right)\right)_{\mathcal{H}}=0, \quad \forall \boldsymbol{w}^h \in V^h . \end{aligned}$$

Therefore we have

$$
\begin{aligned}
&\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}\right),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)\right)_{\mathcal{H}}+\left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)\right)_{\mathcal{H}}\\
&+\left(\mathcal{E}^*\nabla(\varphi_n-\varphi_n^{hk}),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)\right)_{\mathcal{H}}\\
&+j\left(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)-j\left(\boldsymbol{u}_n^{hk},\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)+\left(\int_0^{t_n}\mathcal{M}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\alpha(s))ds\right.\\
&\left.-\sum_{j=1}^{n}k\mathcal{M}\left(t_n-t_{j-1},\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{j-1}^{hk}\right),\alpha_{j-1}^{hk}\right),\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)\right)_{\mathcal{H}}\\
&=\left(\mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}\right),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{w}^h\right)\right)_{\mathcal{H}}+\left(\mathcal{B}\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{w}^h\right)\right)_{\mathcal{H}}\\
&+\left(\mathcal{E}^*\nabla(\varphi_n-\varphi_n^{hk}),\varepsilon\left(\boldsymbol{u}_n-\boldsymbol{w}^h\right)\right)_{\mathcal{H}}\\
&+j\left(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{w}^h\right)-j\left(\boldsymbol{u}_n^{hk},\boldsymbol{u}_n-\boldsymbol{w}^h\right)+\left(\int_0^{t_n}\mathcal{M}(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\alpha(s))ds\right.\\
&\left.-\sum_{j=1}^{n}k\mathcal{M}\left(t_n-t_{j-1},\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{j-1}^{hk}\right),\alpha_{j-1}^{hk}\right),\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n-\boldsymbol{w}^h\right)\right)_{\mathcal{H}},\quad\forall\boldsymbol{w}^h\in V^h.
\end{aligned}
$$

From the property (3.2.23), it is easy to check that

$$
\begin{aligned}
&j\left(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)-j\left(\boldsymbol{u}_n^{hk},\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right)\geq 0\\
&j\left(\boldsymbol{u}_n,\boldsymbol{u}_n-\boldsymbol{w}^h\right)-j\left(\boldsymbol{u}_n^{hk},\boldsymbol{u}_n-\boldsymbol{w}^h\right)\leq c\left\|\boldsymbol{u}_n-\boldsymbol{w}^h\right\|_V\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V.
\end{aligned}
$$

Using the properties (3.2.18)-(3.2.19), (3.2.20)-(3.2.23) and applying the inequality

$$ab\leq\epsilon a^2+\frac{1}{4\epsilon}b^2,\quad a,b,\epsilon\in\mathbb{R},\epsilon>0,\tag{3.3.6}$$

we obtain the following inequality

$$
\begin{aligned}
\left\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right\|_V^2\leq c&\left(\sum_{j=1}^{n}k\left[\left\|\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk}\right\|_V^2+\left\|\alpha_{j-1}-\alpha_{j-1}^{hk}\right\|_{L^2(\Omega)}^2\right]+I_{\mathcal{M},n}^2\right.\\
&\left.+\left\|\boldsymbol{v}_n-\boldsymbol{w}^h\right\|_V^2+\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+\left\|\varphi_n-\varphi_n^{hk}\right\|_W^2\right),\quad\forall\boldsymbol{w}^h\in V^h,
\end{aligned}\tag{3.3.7}
$$

where $I_{\mathcal{M},n}$ is the integration error

$$I_{\mathcal{M},n}=\left\|\int_0^{t_n}\mathcal{M}\left(t_n-s,\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\alpha(s)\right)ds-\sum_{j=1}^{n}k\mathcal{M}\left(t_n-t_{j-1},\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{j-1}\right),\alpha_{j-1}\right)\right\|_{\mathcal{H}}.$$

We utilize (3.2.35) at time $t = t_n$ with the choice $\xi = \xi_n^{hk}$ and add it to (3.3.1) with $\xi^h = \xi_n^h$, after applying some algebraic operations (see [32] for details), in order to estimate $\left\|\alpha_n - \alpha_n^{hk}\right\|_{L^2(\Omega)}$.

$$
\begin{aligned}
&\left\|\alpha_n - \alpha_n^{hk}\right\|_{L^2(\Omega)}^2 + k\sum_{j=1}^{n}\left\|\nabla\left(\alpha_j - \alpha_j^{hk}\right)\right\|_{L^2(\Omega)}^2 \\
&\le c\{\left\|\alpha_0 - \alpha_0^h\right\|_{L^2(\Omega)}^2 + k\sum_{j=1}^{n}\left\|\delta\alpha_j - \dot{\alpha}_j\right\|_{L^2(\Omega)}\left\|\alpha_j - \alpha_j^{hk}\right\|_{L^2(\Omega)} \\
&+ \left\|\alpha_n - \alpha_n^{hk}\right\|_{L^2(\Omega)}\left\|\alpha_n - \xi_n^h\right\|_{L^2(\Omega)} + \left\|\alpha_0 - \alpha_0^{hk}\right\|_{L^2(\Omega)}\left\|\alpha_1 - \xi_1^h\right\|_{L^2(\Omega)} \\
&+ k\sum_{j=1}^{n-1}\left\|\alpha_j - \alpha_j^{hk}\right\|_{L^2(\Omega)}\left\|\left(\alpha_{j+1} - \xi_{j+1}^h\right) - \left(\alpha_j - \xi_j^h\right)\right\|_{L^2(\Omega)} \\
&+ k\sum_{j=1}^{n}\left\|\nabla\left(\alpha_j - \alpha_j^{hk}\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\alpha_j - \xi_j^h\right)\right\|_{L^2(\Omega)} \\
&+ k\sum_{j=1}^{n}\left\|\phi\left(\varepsilon\left(\boldsymbol{u}_j\right), \alpha_j\right) - \delta\alpha_j + \kappa\Delta\alpha_j\right\|_{L^2(\Omega)}\left\|\alpha_j - \xi_j^h\right\|_{L^2(\Omega)} \\
&+ k\sum_{j=1}^{n}\left(\left\|\boldsymbol{u}_j - \boldsymbol{u}_{j-1}^{hk}\right\|_{L^2(\Omega)} + \left\|\alpha_j - \alpha_j^{hk}\right\|_{L^2(\Omega)}\right)\left(\left\|\alpha_j - \alpha_j^{hk}\right\|_{L^2(\Omega)} + \left\|\alpha_j - \xi_j^h\right\|_{L^2(\Omega)}\right)\Bigg\}
\end{aligned}
\tag{3.3.8}
$$

Next, since

$$
\begin{aligned}
\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk} &= \boldsymbol{u}_0 + \int_0^{t^k}\boldsymbol{v}(s)\mathrm{d}s - \boldsymbol{u}_0^h - k\sum_{j=1}^{n-1}\boldsymbol{v}_j^{hk} \\
&= k\sum_{j=1}^{n-1}\left(\boldsymbol{v}_j - \boldsymbol{v}_j^{kk}\right) + \boldsymbol{u}_0 - \boldsymbol{u}_0^h + \sum_{j=1}^{n-1}\left(\int_{t_{j-1}}^{t_j}\boldsymbol{v}(s)\mathrm{d}s - \boldsymbol{v}_j k\right) + \int_{t_{n-1}}^{t_n}\boldsymbol{v}(s)\mathrm{d}s,
\end{aligned}
$$

we have

$$
\left\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right\|_V \le k\sum_{j=1}^{n-1}\left\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right\|_V + \left\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right\|_V + ck\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)},
$$

thus,

$$\left\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\right\|_V^2 \leq c\left\{k \sum_{j=1}^{n-1}\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2+k^2\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}^2\right\}. \tag{3.3.9}$$

We conclude from(3.3.8) that $\dot{\alpha} \in C\left([0,T]; L^1(\Omega)\right)$, hence

$$\left\|\alpha_n-\alpha_{n-1}^{hk}\right\|_{L^2(\Omega)} \leq \left\|\alpha_{n-1}-\alpha_{n-1}^{hk}\right\|_{L^2(\Omega)}+k\|\dot{\alpha}\|_{C([0,T];L^1(\Omega))}. \tag{3.3.10}$$

We combine (3.3.8)-(3.3.10) to obtain

$$\begin{aligned}
&\left\|\alpha_n-\alpha_n^{hk}\right\|_{L^2(\Omega)}^2+k \sum_{j=1}^{n}\left\|\nabla\left(\alpha_j-\alpha_j^{hk}\right)\right\|_{L^2(\Omega)}^2 \\
&\leq c\{\left\|\alpha_0-\alpha_0^h\right\|_{L^2(\Omega)}^2+\left\|\alpha_1-\xi_1^h\right\|_{L^2(\Omega)}^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2 \\
&\quad+k^2\left(\|\boldsymbol{v}\|_{W^{1,1}(0,T;V)}^2+\|\dot{\alpha}\|_{C([0,T];L^1(\Omega))}^2\right)+k \sum_{j=1}^{n}\left\|\delta\alpha_j-\dot{\alpha}_j\right\|_{L^2(\Omega)}^2 \\
&\quad+\left\|\alpha_n-\xi_n^h\right\|_{L^2(\Omega)}^2+k \sum_{j=1}^{n}\left\|\phi\left(\varepsilon\left(\boldsymbol{u}_j\right), \alpha_j\right)-\delta\alpha_j+\kappa\Delta\alpha_j\right\|_{L^2(\Omega)}\left\|\alpha_j-\xi_j^h\right\|_{L^2(\Omega)} \\
&\quad+k \sum_{j=1}^{n}\left\|\nabla\left(\alpha_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2+k^{-1} \sum_{j=1}^{n}\left\|\left(\alpha_{j+1}-\xi_{j+1}^h\right)-\left(\alpha_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2 \\
&\quad+k \sum_{j=1}^{n}\left\|\alpha_j-\alpha_j^{hk}\right\|_{L^2(\Omega)}^2+k \sum_{j=1}^{n-1}\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V^2\Big\}
\end{aligned} \tag{3.3.11}$$

combining (3.3.7), (3.3.11) and (3.3.4), leads us to the following estimate for all $\boldsymbol{w} \in V^h$ and $\{\xi_j^h\}_{j=1}^n \subset \mathcal{K}^h$

$$
\begin{aligned}
&\left\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right\|_V^2+\left\|\alpha_n-\alpha_n^{hk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n}\left\|\nabla\left(\alpha_j-\alpha_j^{hk}\right)\right\|_H^2+\left\|\varphi_n-\varphi_n^{hk}\right\|_V^2\\
&\leq c\left(\sum_{j=1}^{n}k\left[\left\|\boldsymbol{u}_{j-1}-\boldsymbol{u}_{j-1}^{hk}\right\|_V^2+\left\|\alpha_{j-1}-\alpha_{j-1}^{hk}\right\|_{L^2(\Omega)}^2\right]+I_{\mathcal{M},n}^2+\left\|\boldsymbol{v}_n-\boldsymbol{w}^h\right\|_V^2\right.\\
&\quad+\left\|\alpha_0-\alpha_0^h\right\|_{L^2(\Omega)}^2+\left\|\alpha_1-\xi_1^h\right\|_{L^2(\Omega)}^2+\left\|\alpha_n-\xi_n^h\right\|_{L^2(\Omega)}^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2\\
&\quad+k\sum_{j=1}^{n}\left\|\alpha_j-\alpha_j^{hk}\right\|_{L^2(\Omega)}^2+k\sum_{j=1}^{n-1}\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V^2+k\sum_{j=1}^{n}\left\|\nabla(\alpha_j-\xi_j^h)\right\|_{L^2(\Omega)}^2\\
&\quad+k^2\left(\|\boldsymbol{v}\|_{W^{1,1}([0,T];V)}^2+\|\dot{\alpha}\|_{C([0,T];L^1(\Omega))}^2\right)\\
&\quad+\frac{1}{k}\sum_{j=1}^{n}\left\|\left(\alpha_{j+1}-\xi_{j+1}^h\right)-\left(\alpha_j-\xi_j^h\right)\right\|_{L^2(\Omega)}^2\\
&\quad+k\sum_{j=1}^{N}\|\delta\alpha_j-\dot{\alpha}_j\|_{L^2(\Omega)}^2+k\sum_{j=1}^{N}\left\|\Theta\left(\varepsilon\left(\boldsymbol{u}_j\right),\alpha_j\right)\right.\\
&\quad\left.-\frac{1}{k}\left(\alpha_j-\alpha_{j-1}\right)+\kappa\Delta\alpha_j\right\|_{L^2(\Omega)}\cdot\left\|\alpha_j-\xi_j^h\right\|_{L^2(\Omega)}\\
&\quad\left.+\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+\left\|\varphi_n-\Psi^h\right\|_W^2\right),
\end{aligned}
\tag{3.3.12}
$$

where

$$
\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2\leq c\left\{k\sum_{j=1}^{n}\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2\right\}. \tag{3.3.13}
$$

Utilizing 1.3.6, we arrive at the main error estimates listed below

Theorem 3.3.2 *Assume that (3.2.18)-(3.2.26) and the regularity (3.3.13) hold. Let* $(\boldsymbol{u},\alpha,\varphi)\in V\times\mathcal{K}\times W$ *and* $(\boldsymbol{u}^{hk},\alpha^{hk},\varphi^{hk})\in V^h\times\mathcal{K}^h\times W^h$ *be the respective solutions to Problems* PV *and* PV^{hk}*. Then, we have the following error estimates*

For all $\boldsymbol{w}^h=\{\boldsymbol{w}_j^h\}_{j=1}^N\subset V^h$, $\{\alpha_j^h\}_{j=1}^N\subset\mathcal{K}^h$ *and* $\{\psi_j^h\}_{j=1}^N\subset W^h$

$$
\begin{aligned}
&\max_{0\le n\le N}\{\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V^2+\left\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n^{hk}\right\|_V^2+\left\|\alpha_n-\alpha_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\varphi_n-\varphi_n^{hk}\right\|_V^2\}\\
&+k\sum_{j=1}^{n}\left\|\nabla\left(\alpha_j-\alpha_j^{hk}\right)\right\|_H^2\le c\Bigg(\max_{0\le n\le N}I_{\mathcal{M},n}^2\\
&\quad+\max_{0\le n\le N}\{\left\|\boldsymbol{v}_n-\boldsymbol{w}^h\right\|_V^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^{hk}\right\|_V^2+\left\|\alpha_0-\alpha_0^h\right\|_{L^2(\Omega)}^2+\left\|\alpha_1-\alpha_1^h\right\|_{L^2(\Omega)}^2\}\\
&\quad+k\sum_{j=1}^{N}\|\delta\alpha_j-\dot{\alpha}_j\|_{L^2(\Omega)}^2+\max_{0\le n\le N}\left\|\alpha_n-\xi_n^h\right\|_{L^2(\Omega)}^2+\max_{0\le n\le N}\left\|\varphi_n-\Psi^h\right\|_W^2\\
&\quad+k^2\left(\|\boldsymbol{v}\|_{W^{1,1}([0,T];V)}+\|\dot{\alpha}\|_{C([0,T];L^2(\Omega))}\right)\\
&\quad+\max_{0\le n\le N}k\sum_{j=1}^{n}\left\|\nabla(\alpha_j-\xi_j^h)\right\|_{L^2(\Omega)}^2\\
&\quad+\max_{0\le n\le N}\frac{1}{k}\sum_{j=1}^{n}\left\|\left(\alpha_{j+1}-\alpha_{j+1}^h\right)-\left(\alpha_j-\alpha_j^h\right)\right\|_{L^2(\Omega)}^2\\
&\quad+\max_{0\le n\le N}k\sum_{j=1}^{n}\|\Theta\left(\varepsilon\left(\boldsymbol{u}_j\right),\alpha_j\right)-\frac{1}{k}\left(\alpha_j-\alpha_{j-1}\right)+\kappa\Delta\alpha_j\|_{L^2(\Omega)}\cdot\|\alpha_j-\xi_j^h\|_{L^2(\Omega)}\Bigg).
\end{aligned}
\tag{3.3.14}
$$

Corollary

Let's choose the discrete initial conditions $\alpha_0^h=\pi^h\alpha_0$ and $\boldsymbol{u}_0^h=\Pi^h\boldsymbol{u}_0$, where $\pi^h: C(\overline{\Omega})\to\mathcal{K}^h$ is the standard finite element interpolation operator (see, e.g., [18]) and $\Pi^h=(\pi_i^h)_{i=1}^d:[C(\overline{\Omega})]^d\to V^h$. Under the assumptions of Theorem 3.2.1 the following is our estimation of the error

$$
\max_{0\le n\le N}\left\{\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V+\left\|\alpha_n-\alpha_n^{hk}\right\|_{L^2(\Omega)}+\left\|\varphi_n-\varphi_n^{hk}\right\|_V\right\}\le c(h+k). \tag{3.3.15}
$$

Proof.

In the beginning, we must estimate the errors produced by the approximation of the finite element spaces V^h, $\mathcal{K}^h$ and W^h.

We have

$$\max_{0\le n\le N}\inf_{w_n^h\in V^h}\left\|\boldsymbol{u}_n-\boldsymbol{w}_n^h\right\|_V\le ch\|\boldsymbol{u}\|_{C\left([0,T];[H^2(\Omega)]^d\right)},$$

$$\max_{0\le n\le N}\inf_{\xi_n^h\in\mathcal{K}^h}\left\|\alpha_n-\xi_n^h\right\|_{L^2(\Omega)}\le ch\|\alpha\|_{C([0,T];H^2(\Omega))},$$

$$\max_{1\le n\le N}\inf_{\psi_n^h\in W^h}\left\|\varphi_n-\psi_n^h\right\|_W\le ch\|\varphi\|_{C([0,T]:H^2(\Omega))}.$$

We obtain from the definition of the operators π^h and Π^h

$$\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^{hk}\right\|_V\le ch\|\boldsymbol{u}(0)\|_{[H^2(\Omega)]^d}$$

$$\left\|\alpha_0-\alpha_0^h\right\|_{L^2(\Omega)}\le ch^2\left\|\alpha_0\right\|_{H^2(\Omega)}.$$

Finally, using the ideas in [34] we find that the terms corresponding to the damage field are bounded, for example since $\alpha\in H^2(0,T;L^2(\Omega))$,

$$\delta\alpha_j-\dot{\alpha}_j=\frac{1}{k}\int_{t_{j-1}}^{t_j}\left(\dot{\alpha}(t)-\dot{\alpha}\left(t_j\right)\right)\mathrm{d}t=\frac{1}{k}\int_{t_{j-1}}^{t_j}\int_{t_j}^{t}\ddot{\alpha}(s)\mathrm{d}s\,\mathrm{d}t.$$

Thus, it follows $\sum_{j=1}^{N}k\left\|\dot{\alpha}_j-\delta\alpha_j\right\|_{L^2(\Omega)}^2\le ck^2\|\alpha\|_{H^2(0,T;L^2(\Omega))}^2$, also the following estimate was obtained,

$$\frac{1}{k}\sum_{j=1}^{N-1}\left\|\alpha_j-\xi_j^h-\left(\alpha_{j+1}-\xi_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\le ch^2\|\alpha\|_{H^1(0,T;\mathcal{K})}^2.$$

Proceeding as in [17] we find that

$$I_{\mathcal{M},n}\le ck\left(\|\boldsymbol{u}\|_{W^{1,\infty}(0,T;V)}+\|\zeta\|_{H^2(0,T;L^2(\Omega))}\right).$$

■

3.3.4 Numerical Simulations

Some simulation results in test problems including examples in one, two, and three dimensions, have been performed in order to test the accuracy of the numerical method given in the previous section. In this part, we provide a description of the numerical results that demonstrate the algorithm's effectiveness.

The one-dimensional test example

We take into consideration an electro viscoelastic rod $\Omega = (0, L)$ with a fixed left end $x = 0$, and a body force of density $f_o(x,t)$ acting in the x-direction. A gap g exists between its right end $x = L$ and a rigid foundation. The contact is modeled with normal compliance and damage.

The long memory operator has the same form as in ([61])

$$\int_0^t \mathcal{M}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \alpha(s)\right) ds = \int_0^t \eta_*(\alpha)\mathcal{A}\varepsilon\left(\boldsymbol{u}(s)\right)\, ds,$$

where $\eta_* : \mathbb{R} \to \mathbb{R}$ is a function defined by

$$\eta_*(\alpha) = \begin{cases} \alpha_* & if\ \alpha \leq \alpha_* \\ \alpha & if\ \alpha_* < \alpha < 1 \\ 1 & if\ \alpha \geq 1 \end{cases}$$

Thus, the electro-viscoelastic constitutive law with long term-memory and damage 3.2.1 can also be expressed in the rate-type law as follows

$$\dot{\alpha} = \mathcal{B}\varepsilon\left(\dot{\boldsymbol{u}}\right) - \mathcal{E}^* E(\dot{\varphi}) + \eta_*(\alpha)\mathcal{A}\varepsilon\left(\boldsymbol{u}\right)$$

We use here the linear case as follow

$$\dot{\alpha} = a\, \varepsilon\left(\boldsymbol{u}\right) + b\, \varepsilon\left(\dot{\boldsymbol{u}}\right) - c\, E(\dot{\varphi}),$$

where $\varepsilon(\boldsymbol{u}) = \frac{\partial(\boldsymbol{u})}{\partial x}$, while a, b and c are positive material constants, independent of x and t.

In the simulations, the normal compliance function given below is used,

$$p(r) = \frac{1}{\mu} r_+,$$

where $r_+ = \max\{0, r\}$ and μ is a positive constant which represents a deformability coefficient.

We consider the following problem as a first example

Problem $T1D$

Determine a displacement field $\boldsymbol{u} : [0,1] \times [0,1] \to \mathbb{R}$, an electric potentiel $\varphi : [0,1] \times [0,1] \to \mathbb{R}$, and a damage field $\alpha : [0,1] \times [0,1] \to \mathbb{R}$ such that

$$\frac{\partial \sigma(x,t)}{\partial t} = \frac{\partial \boldsymbol{u}(x,t)}{\partial x} + \frac{\partial^2 \boldsymbol{u}(x,t)}{\partial x \partial t} + \frac{\partial^2 \varphi(x,t)}{\partial x \partial t} \qquad \text{in } (0,1) \times (0,1), \tag{3.3.16}$$

$$\boldsymbol{D}(x,t) = \frac{\partial \boldsymbol{u}(x,t)}{\partial x} - 2\,\frac{\partial \varphi(x,t)}{\partial x} \qquad \text{in } (0,1) \times (0,1), \tag{3.3.17}$$

$$\dot{\alpha} - \kappa \Delta \alpha = -3e^{-3t} \qquad \text{in } (0,1) \times (0,1), \tag{3.3.18}$$

$$\frac{\partial \sigma(x,t)}{\partial x} = 2(3e^t - t) \qquad \text{in } (0,1) \times (0,1), \tag{3.3.19}$$

$$\frac{\partial \boldsymbol{D}(x,t)}{\partial x} = 2(e^t - 1) \qquad \text{in } (0,1) \times (0,1), \tag{3.3.20}$$

$$\boldsymbol{u}(0,t) = 0 \qquad \text{for } t \in (0,1), \tag{3.3.21}$$

$$\varphi(0,t) = 0 \qquad \text{for } t \in (0,1), \tag{3.3.22}$$

$$\sigma(1,t) = -\frac{1}{\mu} \max\big(0, \boldsymbol{u}(1,t) - g\big) \qquad \text{for } t \in (0,1), \tag{3.3.23}$$

$$\frac{\partial \alpha}{\partial x}(1,t) = 0 \qquad \text{for } t \in (0,T), \tag{3.3.24}$$

$$\varphi(x,0) = x^2, \quad \boldsymbol{u}(x,0) = 0, \quad \alpha(x,0) = 1 \qquad \text{in } (0,1). \tag{3.3.25}$$

For the numerical computation, the following data have been considered

$$\Omega = (0,1), \qquad \Gamma_1 = \Gamma_a = \{0\}, \qquad \Gamma_2 = \emptyset, \qquad \Gamma_3 = \Gamma_b = \{1\},$$

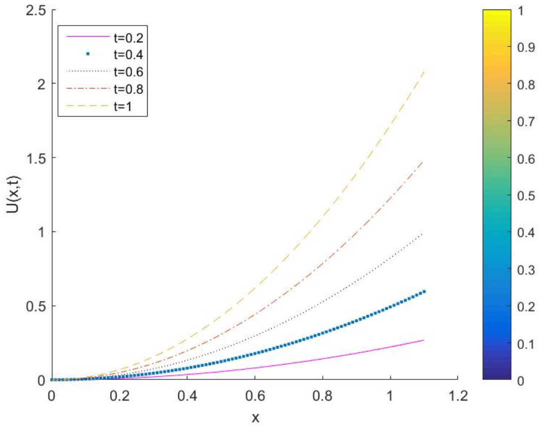
$$T = 1, \qquad a = b = c = 1, \quad \boldsymbol{B} = 2, \qquad g = 0\,m, \quad \mu = 100,$$

$$f_0 = 2(t - 3e^t), \qquad q_0 = 2(e^t - 1), \qquad \Theta\left(\varepsilon(\boldsymbol{u}), \alpha\right) = -3e^{-3t},$$

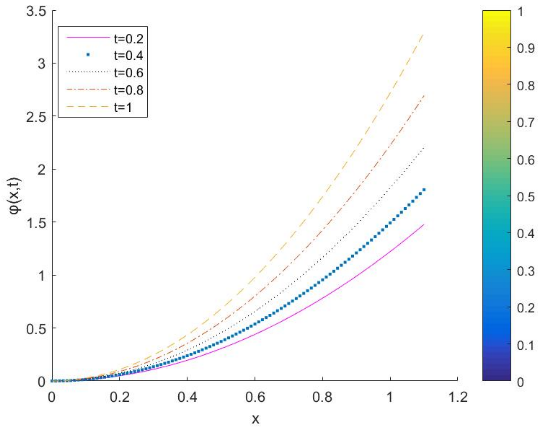
$$\varphi_0(x) = x^2\, N/m, \qquad \boldsymbol{u}_0(x) = 0\, m, \qquad \alpha_0(x) = 1.$$

The fully discrete scheme was implemented by using the spaces V^h, W^h, $\mathcal{K}^h$. We used the discretization parameters $h = k = 0.01$.

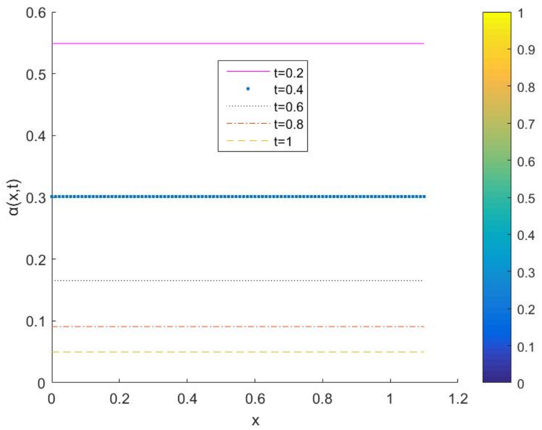
Our numerical results are plotted in Figures 3.1, 3.2, and 3.3. In Figure 3.1 the displacement fields, electric potentials, and the damage function at several times ($t = 0.2, 0.4, 0.6, 0.8, 1$ second) are shown. In Figure 3.2 the displacement, the electric potential and the damage of the right end $x = 1$ are drawn through the time. The evolution of the displacement and the electric potential are plotted in Figure 3.3.



(a) Displacement fields at several times

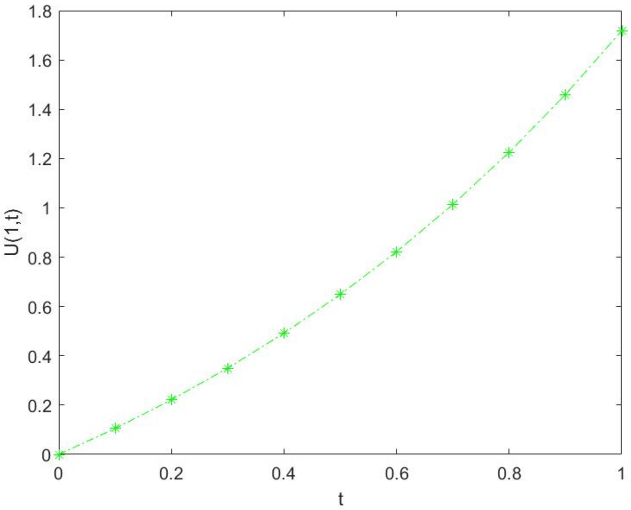


(b) Potential electric fields at several times

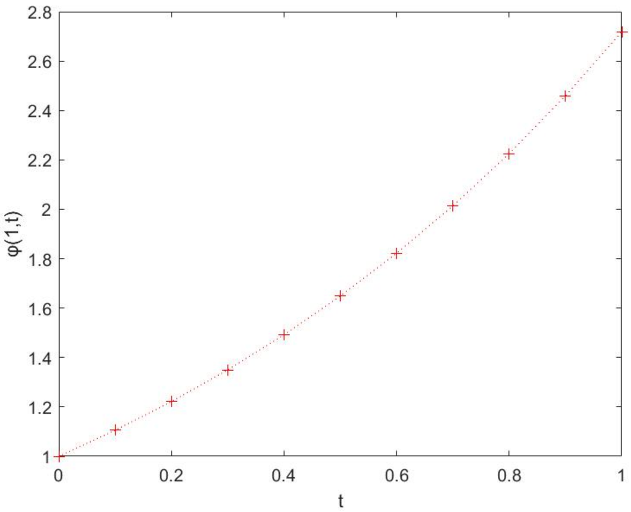


(c) Damage function at several times

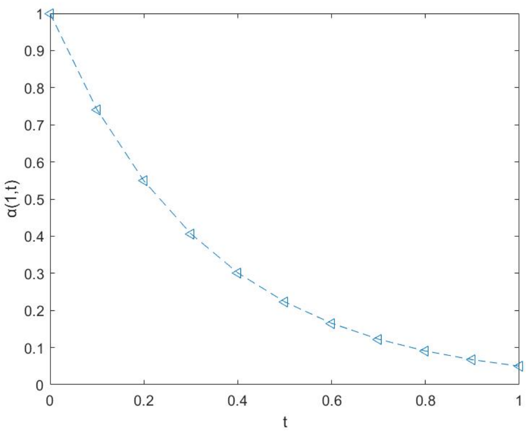
Figure 3.1: Displacements, electric potentials, and damage function at several times



(a) Evolution in time of the displacement of the right end

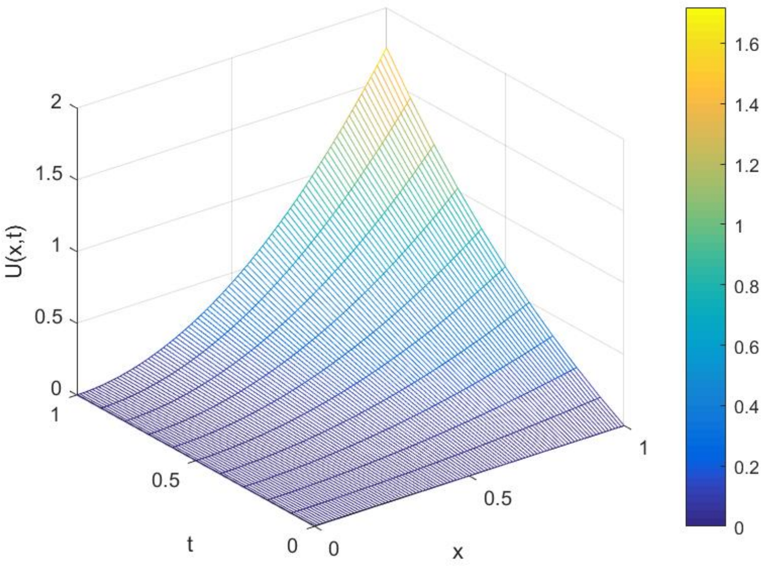


(b) Evolution in time of electric potential of the right end

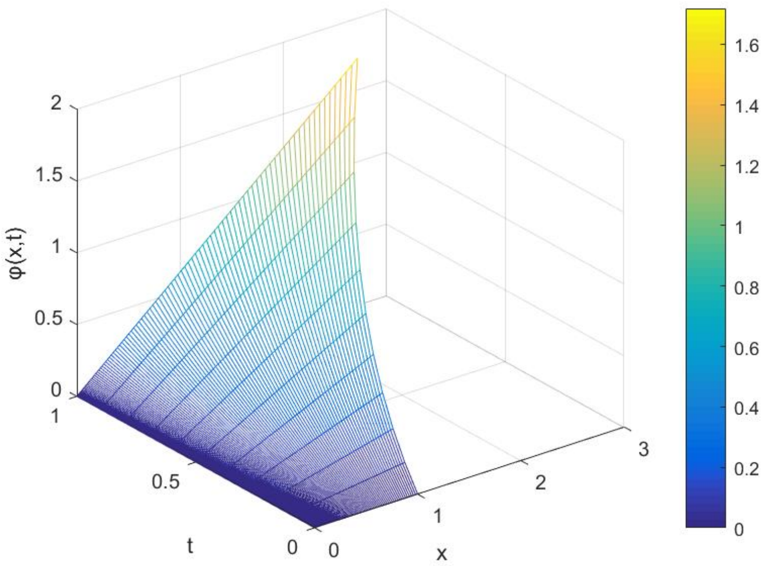


(c) Evolution in time of damage of the right end

Figure 3.2: Evolution in time of the displacement, the electric potential, and the damage of the right end



(a) Evolution of the displacement field



(b) Evolution of the electric potential field

Figure 3.3: Evolution of the displacement and the electric potential fields

General Conclusion

Along this thesis, numerous quasistatic contact problems involving various materials are examined. in the second chapter we have studied two frictional contact problems with normal damped response and damage, The first problem is a thermo-mechanical problem of contact between an elastic-viscoplastic body and a deformable foundation with thermal effects, the second problem is an electro-mechanical problem between two viscoelastic piezoelectric bodies with adhesion. in the third chapter we have studied a frictionless purely mechanical problem with normal compliance and damage, between a viscoelastic piezoelectric body with long memory and a foundation. We used Green's formula to obtain the variational formulation of these problems, and we obtained results of existence and unicity of the weak solution, by using the following arguments: variational equation depending on time, variational equation of evolution, variational inequality of evolution of the parabolic type, differential equation and fixed point. For some cases, we have considered a numerical approximation of the contact problems using a uniform temporal discretization and a spatial discretization by the finite element method. At the end of these discretizations, we showed the existence and uniqueness of the approximate variational problems. Finally, for these schemes, we obtained error estimation results under hypotheses of regularity of the solution. To continue the work accomplished in this thesis, it would be interesting to consider the dynamic contact processes associated with the different contact laws studied. The variational analysis of these problems represents an open research topic that deserves to be addressed in the future. To verify the error estimates obtained in this thesis and to illustrate the behavior of the models studied, simulations and numerical examples based on efficient methods will be welcome. This objective will be realized in further works which will represent a natural continuation of this thesis

Bibliography

[1] A. Aissaoui, *Analyse variationnelle de quelques problemes aux limites de contact avec adhesion et endommagement*, Thèse de doctorat, Université Ferhat Abbes–Setif 1, (2014).

[2] A. Aissaoui, N. Hemici, *A frictional contact problem with damage and adhesion for an electro elastic-viscoplastic body*, Electronic Journal of Differential Equations **11**, (2014), 1–19.

[3] A. Amassad, K.L. Kuttler, M. Rochdi and M. Shillor, *Quasistatic thermoviscoelastic contact problem with slip dependent friction coefficient*, Mathematical and computer modelling, **36** (2002), 839–854.

[4] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[5] A. Azeb Ahmed, S. Boutechebak, *Analysis of a dynamic thermo-elastic-viscoplastic contact problem*, Electron. J. Qual. Theory Differ. Equ , **71** (2013), 1–17.

[6] M. Barboteu, D. Danan, M. Sofonea, *Analysis of a contact problem with normal damped response and unilateral constraint* , ZAMM–Journal of Applied Mathematics and Mechanics, **96** (4) (2016), 408–428.

[7] M. Barboteu, J.R. Fernández, Y. Ouafik, *Numerical analysis of a frictionless viscoelastic piezoelectric contact problem*, M2AN Math. Model. Numer. Anal. (in press)

[8] M. Barboteu, M. Sofonea, *Modelling of piezoelectric contact problems, in Recent Advances in Contact Mechanics*, Springer, Berlin, Heidelberg, (2013), 383–399.

[9] V. BARBU, *Optimal control of variational inequalities*, Vol 100. Pitman Advanced Pub. Program, 1984.

[10] A. R.C. Batra, J.S. Yang, *Saint-Venant's principle in linear piezoelectricity*, Journal of Elasticity **38**, (1995), 209–218.

[11] S. BOUTECHEBAK,*A dynamic problem of frictionless contact for elastic-thermo-viscoplastic materials with damage* , International Journal of Pure and Applied Mathematics, **86** (1) (2013), 173–197.

[12] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Annales de l'institut Fourier, **18** (1) (1968), 115-175.

[13] M. Campo, J.R. Fernández, W. Han, M. Sofonea, *A dynamic viscoelastic contact problem with normal compliance and damage*, Finite Elem. Anal. Des, **42** (1) (2015), 1–24.

[14] O. Chau, B. Awbi, *Quasistatic thermoviscoelastic frictional contact problem with damped response*, Applicable Analysis, **83** (6) (2004), 635–648.

[15] O. Chau, J.R. Fernández , M. Shillor and M. Sofonea , Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion, J. Comput. Appl. Math. 159(2003), 431–465.

[16] O. Chau, M. Shillor and M. Sofonea, Dynamic frictionless contact with adhesion, J. Appl.Math. Phys. (ZAMP) 55 (2004), 32–47.

[17] J. Chen, W. Han, M. Sofonea, *Numerical analysis of a nonlinear evolutionary system with applications in viscoplasticity*, SIAM J. Numer. Anal. 38 (4) (2000) 1171–1199.

[18] P.G. Ciarlet, The finite element method for elliptic problems,North-Holland,Amsterdam, 1978

[19] M. Cocu and R. Rocca, Existence results for unilateral quasistatic contact problems withfriction and adhesion, Math. Model. Num. Anal. 34 (2000), 981–1001.

[20] J. Conway, A Course in Functional Analysis, 2nd edition, Springer, New York, 1990.

[21] A. Djabi, A. Merouani, A. Aissaoui, *A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal effects*, Electronic Journal of Qualitative Theory of Differential Equations, **27** (2015), 1–18.

[22] B. Dupen, *Applied strength of materials for engineering technology*, School of Polytechnic, Purdue University Fort Wayne, (2019).

School of Polytechnic, Purdue University Fort Wayne, 2019.

[23] G. Duvant, J.L Lions, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin, (1976).

[24] C. Eck, J. Jarusek, M. Mircea, *A dynamic elastic-viscoplastic unilateral contact problem with normal damped response and Coulomb friction*, European Journal of Applied Mathematics, **21** (3) (2010), 229–251.

[25] MS. Ferhat, A. Aissaoui, K. Rimi, *Variational analysis for an elastic-thermo- viscoplastic contact problem with normal damped response*, Advances in Mathematics: Scientific Journal, **11** (3) (2022), 131–156.

[26] J.R. Fernández, W.Han, M. Sofonea, J.M. Viaño, *Variational and numerical analysis of a frictionless contact problem for elastic-viscoplastic materials with internal state variables*, The Quarterly Journal of Mechanics and Applied Mathematics, **54** (4) (2001), 501–522.

[27] J.R. Fernández, M. Sofonea , *Variational and numerical analysis of a Signorini's contact problem in viscoplasticity with damage*, Journal of Applied Mathematics, **2** (2) (2003), 87–114.

[28] M. Frémond, Equilibre des structures qui adhèarent à leur support, C. R. Acad. Sci. Paris,Série II 295 (1982), 913—916

[29] M. Frémond, Adhérence des solides, J. Mécanique Théorique et Appliquée, **6** (1987), 383–407.

[30] M. Frémond, Non-Smooth Thermomechanics, Springer, Berlin, (2002).

[31] M. Frémond, B. Nedjar, *Damage, gradient of damage and principle of virtual work* , Int. J. Solids. Stuct, **33** (8) (1996), 1083–1103.

[32] R. Glowinski , *Numerical methods for nonlinear variational problems* ,Springer-Verlag, Berlin, Heidelberg, (1984).

[33] T. Hadj ammar, A. Saïdi, A. Azeb Ahmed, *Dynamic contact problem with adhesion and damage between thermo-electro-elasto-viscoplastic bodies* , Comptes Rendus Mécanique, **345** (5) (2017), 329–336.

[34] W. Han, M. Shillor, M. Sofonea, *Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage*, J. Comput. Appl. Math. 137 (2001), 377-398.

[35] W. Han and M. Sofonea, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, Studies in Advanced Mathematics Vol. 30 (American Mathematical Society–International Press, 2002).

[36] J. Haslinger, I. Hlavacek, and J. Ne´cas, Numerical Methods for Unilateral Problems in Solid Mechanics, in: Handbook of Numerical Analysis, Vol **5** (North-Holland, Amsterdam, 1996), 313—485.

[37] T. Ikeda, *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, 1990.

[38] L. Jianu, M. Shillor, and M. Sofonea, A viscoelastic bilateral frictionless contact problem with adhesion, Applic. Anal, **80**, 2001), 233–255.

[39] N. Kikuchi and J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, (SIAM, Philadelphia, 1988).

[40] A. Klarbring, A. Mikelic, and M. Shillor, Frictional contact problems with normal compliance, Int. J. Eng. Sci. ˇ 26, 811–832 (1988).

[41] A. Kulig, S. Migórski , *Solvability and continuous dependence results for second order nonlinear evolution inclusions with a Volterra-type operator* , Nonlinear Analysis: Theory, Methods & Applications, **75** (13) (2012), 4729–4746.

[42] T. Laursen, Computational Contact and Impact Mechanics (Springer, Netherlands, 2002).

[43] Y. Li and Z. Liu, *Dynamic contact problem for viscoelastic piezoelectric materials with normal damped response and friction*, Journal of Mathematical Analysis and Applications, **373**, (2011), 726-738.

[44] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear inclusions and hemivariational inequalities: models and analysis of contact problems* , Vol 26. Springer Science & Business Media, 2012.

[45] S. Migórski, A. Ochal and M. Sofonea, *Analysis of a piezoelectric contact problem with subdifferential boundary condition*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, **144**, (2014), 1007–1025.

[46] R.D. Mindlin, *Polarisation gradient in elastic dielectrics*, Int. J. Solids Struct. **4**, (1968), 637–642.

[47] J. Necas, I. Hlavácek, *Mathematical theory of elastic and elasto-plastic bodies: an introduction* , Elsevier, 1981.

[48] R.G. Rabel, W. Han, M. Sofonea, *Quasistatic contact problems in viscoelasticity and viscoplasticity* , Vol 30. American Mathematical Soc, 2002.

[49] M. Rochdi, M. Shillor, M. Sofonea,*A quasistatic contact problem with directional friction and damped response*, Applicable Analysis, **68** (3–4) (1998), 409–422.

[50] M. Selmani, L. Selmani, *Frictional contact problem for elastic-viscoplastic materials with thermal effect* , Applied Mathematics and Mechanics, **34** (6) (2013), 761–776.

[51] M. Selmani, L. Selmani, *A frictional contact problem with wear and damage for electro-viscoelastic materials*, Applications of Mathematics **55** (2010), 89–109.

[52] M. Shillor, M. Sofonea, and J. Telega, Models and Variational Analysis of Quasistatic Contact, Lect. Notes Phys. Vol. 655, (Springer, Berlin Heidelberg, 2004).

[53] A. Signorini, Sopra alcune questioni di elastostatica, Atti della Societa Italiana per il Progresso delle Scienze (1933).

[54] M. Sofonea, W. Han, M. Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, *Pure and Applied Mathematics*, 276, Chapman-Hall / CRC Press, New york, 2006.

[55] M. Sofonea, A. Matei, *Mathematical models in contact mechanics*, Vol 398. Cambridge University Press, 2012.

[56] M. Sofonea and A. Matei, Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Note Series, Vol. 398 (Cambridge University Press, Cambridge, 2012).

[57] M. Sofonea and R. Tarraf, *An electro viscoelastic frictional contact problem with damage*, Applicable Analysis, April **86**, (2007), 503-518.

[58] R. A. Toupin , *The elastic dielectrics*, Journal of Rational Mechanics and Analysis **5**, (1956), 849–915.

[59] J. Yang and J. S. Yang, *An introduction to the theory of piezoelectricity*, vol. 9, Springer, 2005.

[60] L. Zhor, Z. Zellagui, H. Benseridia and S. Drablaa, *Variational analysis of an electro viscoelastic contact problem with friction* , Journal of the Association of Arab Universities for Basic and Applied Sciences **14**, (2013), 93–100.

[61] H. Xuan , and X. Cheng, *Numerical analysis and simulation of an adhesive contact problem with damage and long memory* , Journal of the Association of Arab Universities for Basic and Applied Sciences **26** (5), (2021), 2781–2804.

Résumé:

L'objet de cette thèse est la modelisation, l'étude variationnelle et numerique de quelques problèmes de contact avec ou sans frottement , entre un corps deformable et une fondation, ou entre deux corps deformables. Ici nous considérons des lois de comportement non linéaires pour des matériaux elastic viscoplastiques, électro viscoélastiques, et viscoélastiques piezoelectriques avec memoire longue. Pour ces problèmes nous obtenons des formulations variationnelles suivies des résultats d'existence et d'unicité des solutions faibles. Les techniques employées sont basées sur la théorie des opérateurs monotones suivis d'une version du théorème de Cauchy Lipschitz et des arguments du point fixe de Banach. Enfin, nous proposons une approximation numerique du probleme purement mecanique a l'aide de schemas discretises. Pour ces schemas, nous obtenons des resultas d'estimation de l'erreur.

Mots clés: Elasto-viscoplasticité, électro-viscoélasticité, adhésion, endommagement, réponse amortie normale, compliance normale, point fixe, elements finis.

Abstract :

The object of this thesis is the modeling, the variational and numerical study of some contact problems with or without friction, between a deformable body and a foundation, or between two deformable bodies. Here we consider nonlinear laws of behavior for elastic-viscoplastic, electro-viscoelastic, and viscoelastic piezoelectric materials with long memory. For these problems we obtain variational formulations followed by existence and uniqueness results of weak solutions. The techniques employed are based on the theory of monotone operators followed by a version of the Cauchy-Lipschitz theorem and Banach's fixed point arguments. Finally, we propose a numerical approximation of the purely mechanical problem using discrete schemes. For these schemes, we obtain error estimation results.

Key-words: Elasto-viscoplasticity, electro-viscoelasticity, adhesion, damage, normal damped response, normal compliance, fixed point, finite elements.

ملخص:

الهدف من هذه الأطروحة هو النمذجة ، والدراسة المتغيرة والرقمية لبعض مشاكل الاتصال مع أو بدون احتكاك ، بين جسم مشوه وأساس ، أو بين جسمين مشوهين. هنا نأخذ في الاعتبار قوانين السلوك غير الخطية للمواد البلاستيكية المرنة اللزوجة ، والمطاطية الكهربية اللزجة ، والمواد الكهروضغطية اللزجة المطاطية ذات الذاكرة الطويلة. لهذه المشاكل نحصل على صيغ متغيرة تليها نتائج الوجود والتفرد للحلول الضعيفة. تعتمد التقنيات المستخدمة على نظرية المؤثرات الرتيبة متبوعة بنسخة من نظرية كوشي ليبشيتز وحجج النقطة الثابتة لباناخ. أخيرًا ، نقترح تقريبًا عدديًا للمشكلة الميكانيكية البحتة باستخدام مخططات منفصلة. و نحصل على نتائج تقدير الخطأ.

كلمات البحث:

اللدونة المرنة اللزجة، كهرو لزوجة مرنة ، التصاق، تلف ، استجابة مخمدة طبيعية، توافق طبيعي، نقطة ثابتة، عناصر محدودة.