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About Some Caputo-Fabrizio Fractional Differential Equations

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Notations

$\mathbb{N}$	Set of natural numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers.
$\Re e$	Real part.
Γ	Gamma function.
B	Beta function.
$E_{(\alpha,\beta)}$	two-parameter Mittag-Leffler function.
E_{α}	one-parameter Mittag-Leffler function.
$\mathcal{L}$	Laplace Transforms.
$\mathcal{L}^{-1}$	inverse Laplace Transforms.
s	Parameter in the Laplace transformation.
${}_{a}^{RL}D_{x}^{(\alpha)}$	Riemann-Liouville fractional derivative.
${}_{a}^{C}D_{x}^{(\alpha)}$	Caputo fractional derivative.
${}_{a}^{CF}D_{t}^{(\alpha)}$	Caputo-Fabrizio fractional derivative.
I_{a}^{α}	Riemann-Liouville fractional integral.
${}^{CF}I^{\alpha}$	Caputo-Fabrizio fractional integral.
$\mathscr{C}[0,T]$	Space of all continuous functions defined on the interval $[0,T]$.
a.e	almost everywhere .
$\begin{pmatrix} n \\ k \end{pmatrix}$	$C_{n}^{k} = \dfrac{n!}{k!(n-k)!}$.

Introduction

Fractional differential calculus has gained much interest by the many researcher in the last decades and it has strong mathematical background and many papers are attributed to the development of it.Among them, we can cite some for example,[8, 14]. Fractional calculus has been also used for modeling physical phenomena including control systems, mechanics and viscoelasticity.

Several researchers have proposed new definitions of the concept of derivative with fractional order. These definitions go from Riemann–Liouville to the newly proposed one by Caputo and Fabrizio. The old editions of the designed definition of the fractional derivative are a product of convolution of a derivative of a function $f(t)$ with the kernel $k(t,s) = \dfrac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}$ in the case of Caputo old edition between $f(t)$ and the kernel $k(t,s) = \dfrac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}$ in the case of Riemann–Liouville. Both the Riemann–Liouville and Caputo old version are designed with singular kernel. Several researchers have used the derivatives across all fields of sciences and engineering for modeling real-world problems. Recently, a new mathematical design of the concept of derivative with fractional was proposed by Caputo and Fabrizio. The proposed derivative with fractional order is the convolution of first derivative and the function exponential, with this definition, there is no worry of singular kernel and therefore, any model using this new derivative describes the full effect of the memory for the problem.

The aim of this work is to apply the definition of Caputo-Fabrizio to differential equations of fractional order.

This work is divided into five chapter:

- In the first chapter, we will provide some definitions and theorems that we will use in this note.
- In the second chapter, we will mainly introduces definitions and basic properties of fractional derivatives, Riemann-Liouville fractional derivative, Caputo fractional derivative and Caputo-Fabrizio fractional derivative and some of its properties, etc...[5, 14, 11]
- In the third chapter, we will study the existence and uniqueness of solutions of linear fractional differential equations by applying the Laplace transform definition.
  They results in this chapter are taken from they articles [10, 19].
- In the fourth chapter, we will apply some fixed point theorems (Banach's principle of contraction theorem, Krasnoselskii's fixed point theorem) to nonlinear Caputo-Fabrizio fractional differential equations.
- In the last chapter, we will present some results of application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Burgers equation:

  $$\begin{cases} {}^{CF}D_t^{\alpha}u(x,t) = \nu u_{xx} - 2uu_x - \mu u_{xxx} \\ u(x,0) = h(x) \end{cases}$$

  The subject of this chapter is taken from the article [6].

Chaptre 1

Preliminaries

1.1 Some Results from functional analysis

1.1.1 Spaces of Absolutely Continuous and Continuous Functions

Definition 1.1. [4]

Let $\Omega = (a, b)(-\infty \leq a < b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$.

We denote by $L^1(\Omega)$ the space of integrable functions from Ω into $\mathbb{R}$.

$$L^1(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \text{ } is\ measurable\ functions\}$$

$$\| f \|_{L^1} = \| f \|_1 = \int_\Omega | f | \, d\mu = \int | f |$$

Let $p \in \mathbb{R}$ with $1 < p < \infty$, we set

$$L^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, f \text{ } is\ measurable\ functions, | f |^p \in L^1(\Omega)\}$$

with

$$\| f \|_{L^p} = \| f \|_p = \left[\int_\Omega | f(x) |^p d\mu \right]^{\frac{1}{p}}.$$

Definition 1.2. [4]

We set

$$L^\infty(\Omega) = \left\{ \text{f} : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{f is measurable and there is a constant C} \\ \text{suth that } | f | \leq C \ a.e. \ on \ \Omega \end{array} \right\}$$

with

$$\| f \|_{L^\infty} = \| f \|_\infty = \inf\{C; | f(x) | \leq C \ a.e. \ on \ \Omega\}.$$

Definition 1.3. [8]

Let [a,b] a finite interval. We denote by $AC[a,b]$ the space of primitive functions of integrable functions in the sense of Lebesgue

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt, \ \varphi(t) \in L[a,b].$$

and we call $AC[a,b]$ the space of absolutely continuous functions on [a,b].

1.1.2 Sobolev spaces

Consider an open subset Ω of $\mathbb{R}^N$. $D(\Omega)$ is the space of C^∞ ($\mathbb{R}$ or $\mathbb{C}$) functions with compact support in Ω and $D'(\Omega)$ is the space of distributions on Ω. A distribution $T \in D'(\Omega)$ is said to belong to $L^p(\Omega)(1 \leq p \leq \infty)$ if there exists a function $f \in L^p(\Omega)$ such that

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx,$$

Definition 1.4. [20]

Let $m \in \mathbb{N}$ and let $p \in [1, \infty]$. we define

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) \setminus D^\alpha f \in L^p(\Omega) \ \text{ for all } \alpha \in \mathbb{N}^N \text{ such that } \ | \alpha | \leq m\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm :

$$\| f \|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \| D^\alpha f \|_{L^p} .$$

If $p = 2$ one sets $W^{m,2}(\Omega) = H^m(\Omega)$, then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle_{H^m} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u . D^\alpha v dx.$$

And it is equipped with the following norm :

$$\| f \|_{H^m} = \left(\sum_{|\alpha| \leq m} \| D^\alpha u \|^2_{L^2} \right)^{\frac{1}{2}} .$$

1.2 Some real analysis properties

Definition 1.5. (**The continuity**) [17]

Let $f : \mathbb{R} \to \mathbb{R}$ be a application. We say that f is continuous if it is continuous at any point of $\mathbb{R}$.

In other words $f : \mathbb{R} \to \mathbb{R}$ is continuous in a if:

$$\forall a \in \mathbb{R},\ \forall \varepsilon \in \mathbb{R}_+^*,\ \exists \alpha > 0,\ \forall x \in \mathbb{R} : \mid x - a \mid < \alpha \Rightarrow \mid f(x) - f(a) \mid < \varepsilon.$$

Definition 1.6. (**Uniformly continuous applications**) [17]

Let (X,d) and (X',d') metric spaces. A map f: $X \to X'$ is said to be uniformly continue if $\forall \varepsilon \in \mathbb{R}_+^*$, there exists $\alpha \in \mathbb{R}_+^*$ such that

$$\forall (x,y) \in X^2, d(x,y) < \alpha \Longrightarrow d'(f(x), f(y)) < \varepsilon.$$

Definition 1.7. (**Lipschtzian**) [3]

Let G be a part of $\mathbb{R}^2$, $f : G \to \mathbb{R}$ a application and K a positive real number. We say that f is K-Lipschitzian according to y if:

$$\forall t \in G, \forall (y_1, y_2) \in \mathbb{R} \ \ \mid f(t, y_1) - f(t, y_2) \mid \leq K \mid y_1 - y_2 \mid .$$

Definition 1.8. (**Bounded function**)

A function $f : G \subset \mathbb{R} \to \mathbb{R}$ is bounded if:

$$\exists M > 0, \forall t \in G : \mid f(t) \mid \leq M.$$

Definition 1.9. (**Convex function**) [17]

The map f is convex if , $\forall x, y, z \in I \subset \mathbb{R}$ with $x \leqslant y \leqslant z$ for $y = tx + (1-t)z$, we have

$$f(y) \leq tf(x) + (1-t)f(z).$$

Definition 1.10. (**Convolution product**) [14]

The convolution product of two real or complex functions f and g are integrable is:

$$(f * g)(x) = \int_0^x f(x-t)g(t)dt = \int_0^x g(x-t)f(t)dt.$$

Theorem 1.1. (**The derivation under the symbol of integration**) [13]

Assume that :

1. $f : I \subset \mathbb{R} \times [a,b] \to \mathbb{R}$ is continuous,

2. f admits a partial derivative $\dfrac{\partial f}{\partial x}$ continue on I,

3. Applications $u : I \to [a, b]$ and $v : I \to [a, b]$ are derivable,

   then the function

$$\begin{aligned}\varphi :& I \to \mathbb{R}\\ & x \to \int_{u(x)}^{v(x)} f(x,t)dt,\end{aligned}$$

is derivable, of derivative 0

$$\varphi'(x) = \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt + v'(x) f(x, v(x)) - u'(x) f(x, u(x)).$$

Definition 1.11. (**Lebesgue's dominated convergence theorem**) [20]

Let E be a measurable set in $\mathbb{R}$ and let $\{f_n\}$ be a sequence of measurable functions such that

- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ a.e on E .
- For each $n \in \mathbb{N}, \mid f_n(x) \mid \leq g(x)$ a.e on E where g is integrable in the sense of Lebesgue on E.

So

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Theorem 1.2. (**Fubini**) [9]

Let $f(x, y)$ be a summable function over the product of measurable spaces (X, μ) and (Y, ν). Then We have the following assertions:

1) For $\mu-$almost all $x \in X$, the function $f(x, y)$ is summable over Y and its integral over Y is a summable function over X.

2) For $\nu-$ almost all $y \in Y$, the function $f(x, y)$ is summable over X and its integral over X is a summable function over Y.

3) We have:

$$\int_{X\times Y} f(x,y)d(\mu\times\nu)(x,y) = \int_X \Big(\int_Y f(x,y)d\nu(y)\Big) d\mu(x) = \int_Y \Big(\int_X f(x,y)d\mu(x)\Big) d\nu(y).$$

1.3 Some elements of topology

Definition 1.12. (**Norm**)

Let E be a vector space on $\mathbb{R}$. We call a norm on E any application $\| \, . \, \| : E \to \mathbb{R}$ verify

- $\forall x \in \mathbb{E} \ : \| x \|= 0 \Longleftrightarrow x = 0.$
- $\forall \lambda \in \mathbb{R}, \forall x \in \mathbb{E} \quad \| \lambda x \|=| \lambda |\| x \| .$
- $\forall x, y \in \mathbb{E} \ : \| x + y \|\leq\| x \| + \| y \|$ " triangular inequality ".

Exemple 1.1. Space $\mathscr{C}(J, \mathbb{R})$ provided with the norm

$$\| y \|_\infty := \sup\{| y(t) | : \ t \in J\}.$$

Definition 1.13. (**Banach space**) [16]

We call Banach space any space full normalized vector on the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Exemple 1.2. $\mathscr{C}(J, \mathbb{R})$ space of continuous functions on J and with values *in* $\mathbb{R}$ is Banach.

Definition 1.14. (**Open games**) Let E be a metric space. A part A of E is called open if, whenever it contains a point of E, it contains at least one open ball (of radius> 0) having this point as its center

$$(\forall x \in A)(\exists r > 0) : B_0(x, r) \subset A$$

Definition 1.15. (**Closed parties**) [16]

We call closed part of E any part of E whose complement is open.

Exemple 1.3.

Any closed ball is a closed part.

Definition 1.16. (**Compact parts**) [9]

We say that $C \subset \mathbb{R}$ is compact if for any cover of C by open we can extract a finite undercoverage. This translates as follows: if $(U_i)_{i\in I}$ is an open family such that $C \subset (U_i)_{i\in I}$ then there exists a finite subset $J \subset I$, $C \subset \bigcup_{i\in J} U_i$.

Definition 1.17. (**Relatively compact parts**) [17]

We say that A is a relatively compact part of a metric space X if its adhesion is a part compact of X.

Definition 1.18. (**Convex parts**) [20]

Let C be a part of E. We say that C is convex in E if, for all $x, y \in C$ and all $t \in [0,1]$, we have $(1-t)x + ty \in C$.

Definition 1.19. (**Operator**) [2]

Let E be a normalized vector space, a linear application A from E in itself is called a linear operator in E. We call domain of A and we denote it by D_A, where

$$D_A = \{x \in \mathbb{E}, Ax \in \mathbb{E}\}.$$

Definition 1.20. (**Continuous operator**) [2]

The operator A is continuous, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all

$$(x', x'' \in D_A) : \parallel x' - x'' \parallel < \delta \Rightarrow \parallel Ax' - Ax'' \parallel < \varepsilon.$$

Definition 1.21. (**Linear Bounded Operators**) [2]

Let E be a vector space standard, we call a bounded linear operator any continuous linear application of E in E.

• If A is a bounded linear operator, then

$$(\forall x \in D_A) : \parallel Ax \parallel \leq \parallel A \parallel \cdot \parallel x \parallel .$$

where the norm of A being defined by :

$$\parallel A \parallel = \sup_{\|x\| \leq 1, x \neq 0} \parallel Ax \parallel = \sup_{x \in D_A} \frac{\parallel Ax \parallel}{\parallel x \parallel}.$$

Definition 1.22. (**Compact operator**) [9]

Operator A is said to be compact if the image of the set $X \subset \mathbb{R}$ by A that is to say the set A(X) is relatively compact.

1.4 Special functions

1.4.1 The Gamma function

Definition 1.23. [14]

The gamma function $\Gamma(z)$ is defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \ , (\Re e(z) > 0).$$

where $t^{z-1} = e^{(z-1)\log(t)}$, which converges in the right half of the complex plane $\mathfrak{Re}(z) > 0$. Indeed, we have

$$\begin{aligned}\Gamma(x+\mathrm{i}y) &= \int_0^{\infty} e^{-t} t^{x-1+\mathrm{i}y} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} + e^{\mathrm{i}y\log(t)} dt \\ &= \int_0^{\infty} e^{-t} t^{x-1} [\cos(y\log(t)) + \mathrm{i}\sin(y\log(t))] dt.\end{aligned} \tag{1.1}$$

The expression in the square brackets in (1.1) is bounded for all t, convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $x = \mathfrak{Re}(z) > 0$.

Proposition 1.1. [14]

1. $\Gamma(z+1) = z\Gamma(z)$ $(\mathfrak{Re}(z) > 0)$.
2. $\Gamma(n+1) = n!, \ \forall n \in \mathbb{N}$.
3. $\Gamma(1) = 1$.
4. $\Gamma(-m) = \mp\infty, \forall m \in \mathbb{N}$.
5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
6. $\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)...(z+n)}, \ \mathfrak{Re}(z) > 0$.

1.4.2 The Beta function

Definition 1.24. [14]

The Beta function is a type of Euler integral defined by:

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt, \quad (\mathfrak{Re}\ u > 0, \mathfrak{Re}\ v > 0).$$

Proposition 1.2. [14]

The relationship between Euler Beta function and Euler Gamma is given though :

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

1.5 Fixed Point Theorems

Definition 1.25. (**Fixed Point**)

Let T be a map of a set S in itself. We call a fixed point of T any point $s \in S$ such that $T(s) = s$.

Theorem 1.3. (**Banach's principle of contraction**) [3]

Let S be a completed metric space link and let $T : S \to S$ be a contracting application, that is to say there exists $0 < k < 1$ such that

$$d(Tx, Ty) \leqslant k(x, y), \forall x, y \in S.$$

Then T admits a unique fixed point $s \in S$. We have

$$\lim_{n\to\infty} T^n(s) = s,$$

with

$$d(T^n(s), s) \leq \frac{k^n}{1-k} d(s, T(s)).$$

Proof. See [3] □

Theorem 1.4. (**Arzela-Ascoli**) [9]

Let $C(X)$ be the vector normalized space of real functions continuous on a compact metric space X with norm:

$$\| f \| = \sup_{x\in X} | f(x) | .$$

For a family $A \subset C(X)$ is relatively compact, if and only and if A is:

• **Uniformly bounded**:

$$\exists \, C : |f(x)| \leq C, \forall f \in A, \forall x \in X,$$

• **Equicontinuous**:

$$\forall \varepsilon > 0, \exists \, \delta > 0, |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon, \forall f \in A.$$

Proof. See [9] □

Theorem 1.5. (**Krasnoselskii's fixed point theorem**) [7]

Let $X \in E$ nonempty subset of E and $f, g : X \to E$ such that

- X : be a closed, convex.

- E : Banach space.
- f *and* g are continuous, f is compact, g a contraction and $f(X) + g(X) \subseteq X$.

Then $f + g$ admits a fixed point in X.

1.6 Laplace transforms

Let us recall some basic tools of the Laplace transform

Definition 1.26. [14]

The Laplace transform is a practical method for solving differential equations and differential systems, let f be a function defined for all the variable. $x > 0$

• Laplace transform of $f(x)$ is defined by:

$$F(s) = \mathcal{L}[f(x)](s) = \int_0^{+\infty} f(x)e^{-sx}dx, \quad s \in \mathbb{C}.$$

• The original $f(x)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform,

$$f(x) = \mathcal{L}^{-1}[F(s)](x) = \int_{c-i\infty}^{c+i\infty} F(s)e^{sx}ds, \quad c = \Re e(s) > c_0$$

• Laplace transform of the convolution

$$\mathcal{L}[f(x) * g(x)](s) = F(s).G(s).$$

We assume that $F(s)$ and $G(s)$ exist.

Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order n of the function $f(x)$:

$$\mathcal{L}[f^n(x)](s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^k f^{(n-k-1)}(0).$$

Table summarizes some Laplace transformations of some functions and some properties of Laplace transform

The function	Transforme	The function	Transforme
$x^{m-1}\mathrm{e}^{ax}$	$\dfrac{\Gamma(m)}{(s-a)^m} \quad (m>0)$	$af(x)+bg(x)$	$aF(s)+bG(s)$
$\cos\beta x$	$\dfrac{s}{s^2+\beta^2}$	$\underbrace{\int_0^x \mathrm{d}t \cdots \int_0^t f(t')\,\mathrm{d}t'}_{\text{n fois}}$	$s^{-n}F(s)$
$\sin\beta x$	$\dfrac{\beta}{s^2+\beta^2}$	$f^n(x)$	$s^nF(s)-\sum_{j=0}^{n-1} s^{n-1-j}f^j(0)$
$x^m \quad (m>-1)$	$\dfrac{\Gamma(m+1)}{s^{m+1}}, \quad \Re e \quad s>0$	$f(cx)$	$\dfrac{1}{c}F(s/c)$
$\delta(x-a)$	e^{-as}	$xf(x)$	$-\dfrac{dF(s)}{df}$
$H(x-a)$	$\dfrac{1}{s}e^{-as}$	$\dfrac{f(x)}{x}$	$\int_s^\infty F(s)\,(s')\,\mathrm{d}s'$
$(\pi x)^{\frac{1}{2}}\mathrm{e}^{-a^2/4x}$	$\dfrac{1}{\sqrt{s}}\mathrm{e}^{-a\sqrt{s}}$	$\int_0^x g(x-t)f(t)\mathrm{d}t$	$F(s)G(s)$

Chaptre 2

Fractional calculus

2.1 Riemann-Liouville and Caputo fractional derivatives

2.1.1 Riemann-Liouville fractional integrals

The definition of fractional integral in the Riemann-Liouville sense is a generalization, of the Cauchy formula (1789 - 1857), which is obtained as follows: [12]

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, we denote

$$I^1 f(x) = \int_a^x f(t)dt.$$

Double integration

$$I^2 f(x) = \int_a^x \int_a^t f(\mu)d\mu dt = \int_a^x (x-t)f(t)dt.$$

By repeating the process (n - 1) times, we obtain the following relation:

$$I^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt.$$

For all $n \in \mathbb{N}$ where the generalization of the factorial by the Gamma function:

$$(n-1)! = \Gamma(n).$$

Definition 2.1. [12]

The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$, for a function $f \in \mathscr{C}([a, b])$ is defined by:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$

Proposition 2.1.

1. $I_a^\alpha(I_a^\beta f)(x) = I_a^\beta(I_a^\alpha f)(x) = I_a^{\alpha+\beta} f(x)$.

2. Let A,B are fixed elements of the body $\mathbb{R}$ or $\mathbb{C}$, $I_a^\alpha[Af(x)+Bg(x)] = AI_a^\alpha f(x)+BI_a^\alpha g(x)$.

2.1.2 Riemann-Liouville fractional derivative

Definition 2.2. [15]

The popular definition of fractional derivative is this one:

$$^{RL}D_a^{(\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)dt,$$

$$(n-1 \leqslant [\Re e(\alpha)] < n), \ \ and \ x > a.$$

Proposition 2.2. Riemann-Liouville operator has the following important properties:

• ${}_a^{RL}D_t^{(\alpha)}\ {}_a^{RL}D_t^{(\beta)} f(x) = {}_a^{RL}D_t^{(\alpha+\beta)} f(x)$.

• For $\alpha = m \in \mathbb{N}$ we have:

$$^{RL}D_a^{(0)} f(x) = \frac{1}{\Gamma(1)}\left(\frac{d}{dx}\right)\int_a^x f(t)dt = f(x),$$

$$^{RL}D_a^{(m)} f(x) = \frac{1}{\Gamma(1)}\left(\frac{d^{m+1}}{dx^{m+1}}\right)\int_a^x f(t)dt = \frac{d^m}{dx^m} f(x),$$

consequently the fractional derivative in the sense of Riemann-Liouville coincides with the derivative classic by $\alpha \in \mathbb{N}$.

Remark 2.1.

$$^{RL}D_a^{(\alpha)} f(x) = \left(\frac{d}{dx}\right)^n (I_a^{n-\alpha} f)(x),$$

such that : $n = [\Re e(\alpha)] + 1, x > a$.

2.1.3 Caputo fractional derivative

Although fractional derivation in the sense of Riemann-Liouville played an important role in the development of fractional calculus, because of its applications in pure and applied mathematics. However, given that the derivative in the sense of Riemann-Liouville of a constant is not zero and that the initial conditions of the Cauchy problem are expressed by fractional order derivatives, Caputo offers another approach where the derivative of the constant is zero and the initial conditions are expressed as in the classical case by whole order derivatives.

Definition 2.3. [15]

Let $0 < n-1 < \alpha < n$ and f be a function of class $C^n([a,b])$. The fractional derivative of order α in the sense of Caputo of the function f is defined by:

$$^{C}D^{(\alpha)}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x f^n(t)(x-t)^{n-\alpha-1}dt.$$

• Under natural conditions on the function $f(x)$, for $\alpha \to n$ the Caputo derivative becomes a conventional $n-th$ derivative of the function $f(x)$.

Indeed let us assume that $0 \leq n-1 < \alpha < n$ and that the function $f(x)$ has $n+1$ continuous bounded derivatives in $[a,T]$ for every $T > a$ then[14]

$$\lim_{\alpha\to n} {}_a^C D_x^{(\alpha)} f(x) = \lim_{\alpha\to n}\left(\frac{1}{\Gamma(n-\alpha)}\int_a^x f^n(t)(x-t)^{n-\alpha-1}dt\right),$$

from integration by parts we have

$$\begin{aligned}\lim_{\alpha\to n} {}_a^C D_x^{(\alpha)} f(x) &= \lim_{\alpha\to n}\left(\frac{f^{(n)}(a)(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)}\int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt\right)\\ &= f^n(a) + \int_a^x f^{(n+1)}(t)dt\\ &= f^{(n)}(x), \quad n = 1, 2, \ldots\end{aligned}$$

• Non-commutation [14]

$${}_a^C D_x^{(\alpha)}\left({}_a^C D_x^{(m)} f(x)\right) = {}_a^C D_x^{(\alpha+m)} f(x) \neq {}_a^C D_x^{(m)}({}_a^C D_x^{(\alpha)} f(x)), \quad (m = 0, 1, 2, \ldots, \quad n-1 < \alpha < n).$$

The interchange of the differentiation operators in formulas is allowed under different conditions:

$${}_a^C D_x^{(\alpha)}\left({}_a^C D_x^{(m)} f(x)\right) = {}_a^C D_x^{(m)}\left({}_a^C D_x^{(\alpha)} f(x)\right) = {}_a^C D_x^{(\alpha+m)} f(x),$$

$$f^{(s)}(0) = 0, \quad s = n, n+1, \ldots, m,$$

$$(m = 0, 1, 2, \ldots, n-1 < \alpha < n).$$

Exemple 2.1. Consider the function:

$$f(x) = x^\beta.$$

for $0 < n-1 < \alpha < n$, we have:

$$^{C}D^{(\alpha)}f(x) = I^{n-\alpha}(D^n x^\beta),$$

or

$$D^n x^\beta = \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)} x^{\beta-n}\right).$$

As a result:

$$I^{n-\alpha}\left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}x^{\beta-n}\right)=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}t^{\beta-n}dt,$$

by performing the change of variable $t=yx$ so $dt=xdy$, we obtain:

$$\begin{aligned}\int_0^x(x-t)^{n-\alpha-1}t^{\beta-n}dt&=\int_0^1(x-xy)^{n-\alpha-1}(xy)^{\beta-n}xdy\\&=\int_0^1 x^{n-\alpha-1}(1-y)^{n-\alpha-1}y^{\beta-n}x^{\beta-n+1}dy\\&=\int_0^1 x^{\beta-\alpha}(1-y)^{n-\alpha-1}y^{\beta-n}dy\\&=x^{\beta-\alpha}\int_0^1(1-y)^{n-\alpha-1}y^{\beta-n}dy\\&=x^{\beta-\alpha}B(n-\alpha,\beta-n+1)\\&=x^{\beta-\alpha}\frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}.\end{aligned}$$

So

$$I^{n-\alpha}\left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}x^{\beta-n}\right)=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}\frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(\beta-n+1)}x^{\beta-\alpha},$$

finally,we obtain

$${}^{C}D^{(\alpha)}x^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}.$$

In particular, for $\beta=0$, we have:

$${}^{C}D^{(\alpha)}x^{0}=D^{(\alpha)}1=0.$$

unlike the Riemann-Liouville derivative, the fractional order derivative in the Caputo sense of a constant is zero.

2.1.4 Some properties of fractional derivatives

Theorem 2.1. (**Linearity**) [14]

Similarly to integer-order differentiation, fractional differentiation is a linear operator:

$$D^{(\alpha)}(\lambda f(x)+\mu g(x))=\lambda D^{(\alpha)}f(x)+\mu D^{(\alpha)}g(x).$$

Proof.

For example, if $D^{(\alpha)}$ is the Caputo operator (where $n-1 \leq \alpha < n$ and $n=1$), by definition we have:

$$\begin{aligned}
{}_a^C D_x^{(\alpha)}(\lambda f(x) + \mu g(x)) &= \frac{1}{\Gamma(1-\alpha)} \int_a^x (\lambda f(t) + \mu g(t))'(x-t)^{-\alpha} dt \\
&= \frac{1}{\Gamma(1-\alpha)} \int_a^x (\lambda f'(t) + \mu g'(t))(x-t)^{-\alpha} dt \\
&= \frac{\lambda}{\Gamma(1-\alpha)} \int_a^x f'(t)(x-t)^{-\alpha} dt + \frac{\mu}{\Gamma(1-\alpha)} \int_a^x g'(t)(x-t)^{-\alpha} dt \\
&= \lambda {}_a^C D_x^{(\alpha)} f(x) + \mu {}_a^C D_x^{(\alpha)} g(x).
\end{aligned}$$

□

Theorem 2.2. (**The Leibniz Rule**) [14]

For all $n \in \mathbb{N}$ we have

$$\frac{d^n}{dt^n}(f(x)g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

The generalization of this formula gives us

$$D^{(\alpha)}(f(x)g(x)) = \sum_{k=0}^{n} \binom{\alpha}{k} f^{(k)}(x) D^{(\alpha)} g^{(\alpha-k)}(x) + R_n^{\alpha}(x).$$

or $n \geqslant \alpha + 1$ and

$$R_n^{\alpha}(x) = \frac{1}{n!\Gamma(-\alpha)} \int_a^t (x-s)^{-\alpha-1} g(s) ds \int_s^t f^{(n+1)}(\xi) d\xi,$$

with

$$\lim_{n\to\infty} R_n^{\alpha}(x) = 0.$$

If f and g are continuous in [a,t] as well as all their derivatives, the formula becomes:

$$D^{\alpha}(f(x)g(x)) = \sum_{k=0}^{n} \binom{\alpha}{k} f^{(k)}(x) D^{\alpha} g^{(\alpha-k)}(x).$$

where D^{α} is the fractional derivative in the sense of Riemann-Liouville.

2.2 Caputo-Fabrizio fractional derivative (CFFD)

Because of the singularity in the kernel of the Caputo fractional derivative[8, 14] at the end point of the interval of integration, the Caputo fractional derivative is not always a suitable kernel to accurately describe the memory effect in a real system. Caputo and Fabrizio[5] has recently proposed a new fractional derivative without any singularity in its kernel. The kernel of the new fractional derivative has the form of an exponential function. More recently, Losada and Nieto[10] derived the fractional integral associated with the new fractional Caputo–Fabrizio fractional derivative.

This section is devoted to studying the basic definitions and results about the Caputo-Fabrizio fractional derivative.

Let us recall the usual Caputo fractional time derivative (CFD) of order α, is given by :

$$ {}^{C}D_t^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(\tau)(t-\tau)^{-\alpha} d\tau. \tag{2.1}$$

Definition 2.4. [5]

Let $f \in H^1(a,b), b > a, \alpha \in [0,1]$ the Caputo-Fabrizio fractional derivative is defined as

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\alpha \frac{(t-\tau)}{1-\alpha}\right] d\tau, \tag{2.2}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

If the function does not belong to $H^1(a,b)$ then, the derivative can be reformulated as

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t (f(t) - f(\tau)) \exp\left[-\alpha \frac{(t-\tau)}{1-\alpha}\right] d\tau.$$

The definition of the CFFD was improved by Losada and Nieto to become [10]

$$ {}^{CF}D_t^{(\alpha)} f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_a^t f'(\tau) \exp\left[-\alpha \frac{(t-\tau)}{1-\alpha}\right] d\tau.$$

Now, it is worth to observe that if we put [5]

$$ \sigma = \frac{1-\alpha}{\alpha} \in [0,\infty] \quad , \alpha = \frac{1}{1+\sigma} \in [0,1].$$

the definition (2.4) of CFFD assumes the form

$$ \tilde{D}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau.$$

where $\sigma \in [0, \infty]$ and $N(\sigma)$ is the corresponding normalization term of M(α), such that $N(0) = N(\infty) = 1$. Moreover because

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau),$$

and for $\alpha \to 1$, we have $\sigma \to 0$

$$\begin{aligned}\lim_{\alpha \to 1} {}^{CF}D_t^{(\alpha)} f(t) &= \lim_{\alpha \to 1} \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\ &= \lim_{\sigma \to 0} \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f'(t).\end{aligned}$$

Otherwise, when $\alpha \to 0$, then $\sigma \to +\infty$. Hence,

$$\begin{aligned}\lim_{\alpha \to 0} {}^{CF}D_t^{(\alpha)} f(t) &= \lim_{\alpha \to 0} \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \\ &= \lim_{\sigma \to +\infty} \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f(t) - f(a).\end{aligned}$$

Let us consider the (CFFD) of a particular function, as $f(t) = \sin \omega t$, for $\alpha = 0.66, a = -8$ and $\omega = 1$

$${}^{CF}D_t^{0.66} \sin \omega t = \frac{M(0.66)}{0.33} \int_a^t \cos \tau \exp -2(t-\tau) d\tau.$$

The simulation of this derivative produces the following pictures.

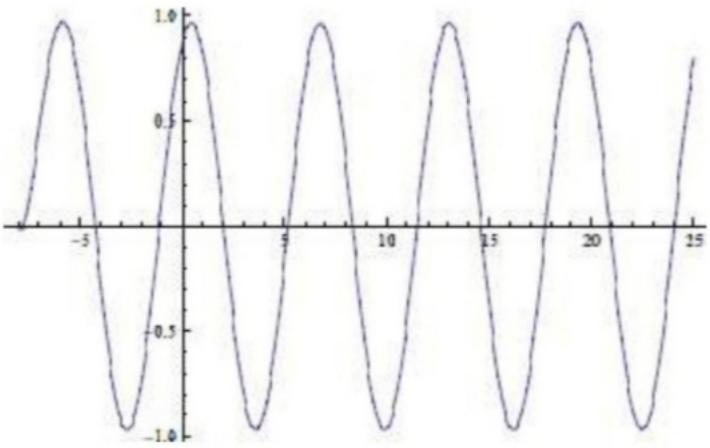

Figure 2.1: Simulation of (CFFD), with $\alpha = 0.66$

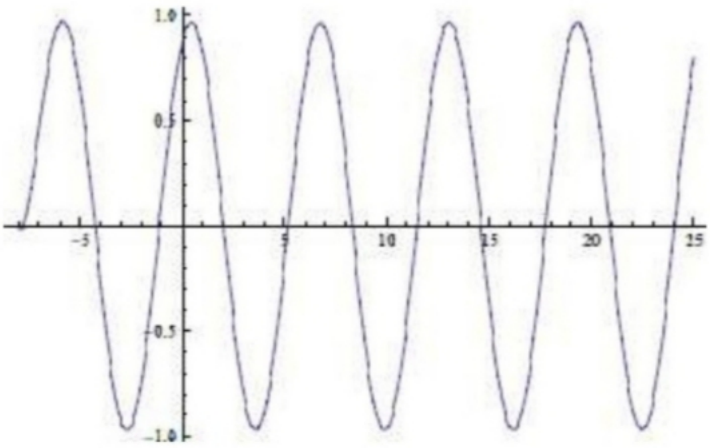

Figure 2.2: Simulation of (CFD), with $\alpha = 0.66$

From these two simulations with (CFFD) $\alpha = 0.66$, it appears as the classical is very similar to the (CFD). Otherwise, when we study models with α close to 0, we see a different behavior.

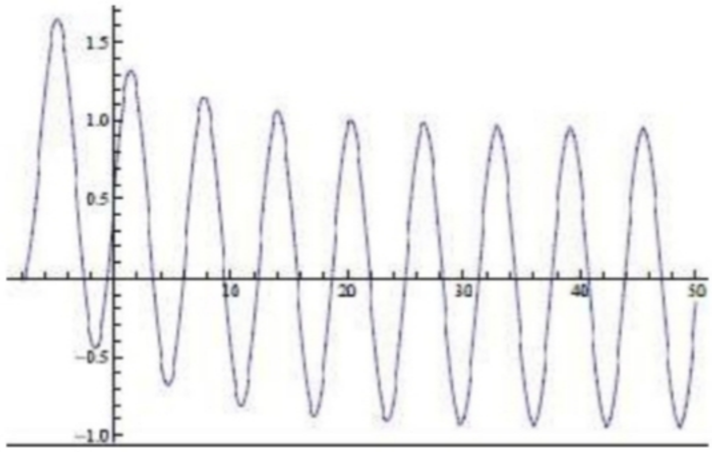

Figure 2.3: Simulation of (CFFD), with $\alpha = 0.1$

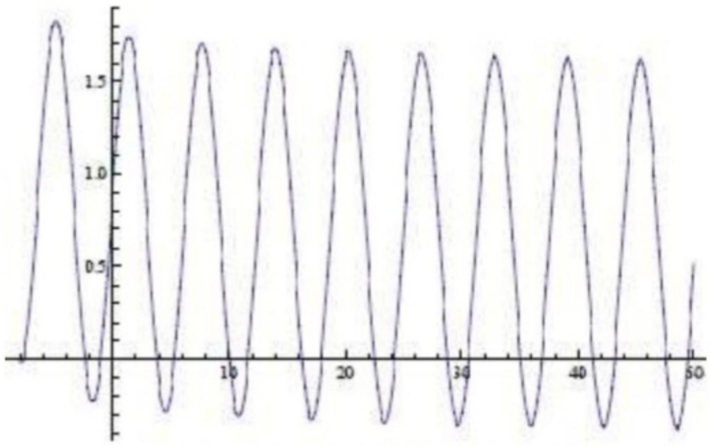

Figure 2.4: Simulation of (CFD), with $\alpha = 0.1$

So that, for $\alpha = 0.1$ in Figure 2.3 and Figure 2.4 we observe different actions between (CFFD) and (CFD) simulations. In particular the classical (CFD) is more affected by past, compared with the (CFFD) which show a rapid stabilization.

If $n \geq 1$, and $\alpha \in [0, 1]$ the fractional time derivative ${}^{CF}D_t^{(\alpha+n)} f(t)$ of order $(n + \alpha)$ is defined by:

$$ {}^{CF}D_t^{(\alpha+n)} f(t) :={}^{CF} D_t^{(\alpha)}({}^{CF}D_t^{(n)} f(t)). \tag{2.3}$$

Theorem 2.3. [5]

For (CFFD), if the function f (t) is such that

$$f^{(s)}(a) = 0, \quad s = 1, 2, ..., n$$

then, we have

$$ {}^{CF}D_t^{(\alpha)}({}^{CF}D_t^{(n)} f(t)) ={}^{CF} D_t^{(n)}({}^{CF}D_t^{(\alpha)} f(t)).$$

Proof.

We begin considering $n=1$, then from definition(2.3) of ${}^{CF}D_t^{(\alpha+1)}f(t)$, we obtain

$$ {}^{CF}D_t^{(\alpha)}({}^{CF}D_t^{(1)}f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t f''(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau. $$

Hence, after an integration by parts and assuming $f'(a)=0$, we have

$$ \begin{aligned} {}^{CF}D_t^{(\alpha)}({}^{CF}D_t^{(1)}f(t)) &= \frac{M(\alpha)}{1-\alpha}\int_a^t (\frac{d}{d\tau}f'(\tau))\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau \\ &= \frac{M(\alpha)}{1-\alpha}\left[\int_a^t \frac{d}{d\tau}f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau \right. \\ &\left. -\frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right] \\ &= \frac{M(\alpha)}{1-\alpha}\left[f'(t)-\frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right]. \end{aligned} $$

otherwise

$$ \begin{aligned} {}^{CF}D_t^{(1)}({}^{CF}D_t^{(\alpha)}f(t)) &= \frac{d}{dt}(\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau) \\ &= \frac{M(\alpha)}{1-\alpha}\left[f'(t)-\frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right]. \end{aligned} $$

For $n=2$ we obtain

$$ \begin{aligned} {}^{CF}D_t^{(\alpha)}({}^{CF}D_t^{(2)}f(t)) &= \frac{M(\alpha)}{1-\alpha}\int_a^t (\frac{d}{d\tau}f''(\tau))\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau \\ &= \frac{M(\alpha)}{1-\alpha}\left[f''(t)-\frac{\alpha}{1-\alpha}\int_a^t f''(\tau)\exp \frac{-\alpha(t-\tau)}{1-\alpha}d\tau\right] \\ &= \frac{M(\alpha)}{1-\alpha}\left[f''(t)-\frac{\alpha}{1-\alpha}f'(t)+\left(\frac{\alpha}{1-\alpha}\right)^2\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}\right]. \end{aligned} $$

Otherwise

$$ \begin{aligned} {}^{CF}D_t^{(2)}({}^{CF}D_t^{(\alpha)}f(t)) &= \frac{d^2}{dt^2}(\frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau) \\ &= \frac{d}{dt}(\frac{M(\alpha)}{1-\alpha})\left[\frac{d}{dt}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau)\right] \\ &= \frac{d}{dt}(\frac{M(\alpha)}{1-\alpha})\left[f'(t)-\frac{\alpha}{1-\alpha}\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau)\right] \\ &= \frac{M(\alpha)}{1-\alpha}\left[f''(t)-\frac{\alpha}{1-\alpha}f'(t)+\left(\frac{\alpha}{1-\alpha}\right)^2\int_a^t f'(\tau)\exp -\frac{\alpha(t-\tau)}{1-\alpha}d\tau\right]. \end{aligned} $$

So

$$^{CF}D_t^{(\alpha)}(^{CF}D_t^{(1)}f(t)) =^{CF} D_t^{(1)}(^{CF}D_t^{(\alpha)}f(t)).$$

$$^{CF}D_t^{(\alpha)}(^{CF}D_t^{(2)}f(t)) =^{CF} D_t^{(2)}(^{CF}D_t^{(\alpha)}f(t)).$$

$$^{CF}D_t^{(\alpha)}(^{CF}D_t^{(3)}f(t)) =^{CF} D_t^{(3)}(^{CF}D_t^{(\alpha)}f(t)).$$

Can be generalized for any $n > 0$,

$$^{CF}D_t^{(\alpha)}(^{CF}D_t^{(n)}f(t)) =^{CF} D_t^{(n)}(^{CF}D_t^{(\alpha)}f(t)).$$

□

Proposition 2.3. (**Linearity**)

Let $^{CF}D^{(\alpha)}$ Caputo-Fabrizio operator satisfy

$$^{CF}D^{(\alpha)}(\lambda f(x) + \mu g(x)) = \lambda^{CF}D^{(\alpha)}f(x) + \mu^{CF}D^{(\alpha)}g(x).$$

Proof. According to the definition(2.4), we have

$$\begin{aligned}
{}_a^{CF}D_t^{(\alpha)}(\lambda f(t) + \mu g(t)) &= \frac{M(\alpha)}{(1-\alpha)} \int_a^t (\lambda f(\tau) + \mu g(\tau))' \exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \frac{M(\alpha)}{(1-\alpha)} \int_a^t (\lambda f'(\tau) + \mu g'(\tau)) \exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \frac{\lambda M(\alpha)}{(1-\alpha)} \int_a^t f'(\tau) \exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right] d\tau + \frac{\mu M(\alpha)}{(1-\alpha)} \int_a^t g'(\tau) \exp\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right] d\tau \\
&= \lambda \, {}_a^{CF}D_t^{(\alpha)} f(t) + \mu \, {}_a^{CF}D_t^{(\alpha)} g(t).
\end{aligned}$$

□

2.2.1 Laplace transform of the CFFD

Definition 2.5. [5]

It is well known that Laplace Transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition (CFFD) with $a = 0$ the Laplace Transform becomes like this, for $0 < \alpha < 1$

$$\mathcal{L}[^{CF}D_t^{(\alpha)}f(t)](s) = \frac{s\mathcal{L}[f(t)](s) - f(0)}{2(s + \alpha(1-s))}, \quad s > 0. \tag{2.4}$$

Lemma 2.1. [5]

The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0, 1)$ and $n \in \mathbb{N}$ is given by :

$$\mathcal{L}[^{CF}D_t^{(\sigma)}f(t)](s) = \frac{s^{n+1}\mathcal{L}[f(t)](s) - s^n f(0) - s^{n-1}f'(0) - ... - f^{(n)}(0)}{s + \alpha(1-s)}. \tag{2.5}$$

Proof.

We apply the Laplace Transform to definition (2.4), we suppose the function $M(\alpha) = 1$,

$$\mathcal{L}[{}^{CF}D_t^{(\alpha)} f(t)](s) = \frac{1}{1-\alpha} \int_0^\infty \exp(-st) \int_0^t f'(\tau) \exp -\frac{\alpha(t-\tau)}{1-\alpha} d\tau dt.$$

Hence, from the property of Laplace transform of a convolution, we have :

$$\begin{aligned}
\mathcal{L}[{}^{CF}D_t^{(\alpha)} f(t)](s) &= \frac{1}{1-\alpha} \mathcal{L}[f'(t)] \mathcal{L}[\exp -\frac{\alpha t}{1-\alpha}] \\
&= \frac{1}{1-\alpha} \Big(\int_0^\infty f'(t) \exp(-st) dt \Big) \Big(\int_0^\infty \exp(-\frac{\alpha t}{1-\alpha}) \exp(-st) dt \Big) \\
&= \frac{1}{1-\alpha} \Big(- f(0) + s \int_0^\infty f(t) \exp(-st) dt \Big) \Big(\int_0^\infty \exp(-t(s + \frac{\alpha}{1-\alpha})) dt \Big) \\
&= \Big(s\mathcal{L}[f(t)] - f(0) \Big) \Big(\frac{1}{s + \alpha(1-s)} \Big) \\
&= \frac{s\mathcal{L}[f(t)] - f(0)}{s + \alpha(1-s)}.
\end{aligned}$$

in the same way

$$\begin{aligned}
\mathcal{L}[{}^{CF}D_t^{(\alpha+1)} f(t)](s) &= \frac{1}{1-\alpha} \mathcal{L}[f''(t)] \mathcal{L}[\exp -\frac{\alpha t}{1-\alpha}] \\
&= \frac{1}{1-\alpha} \Big(\int_0^\infty f''(t) \exp(-st) dt \Big) \Big(\int_0^\infty \exp(-\frac{\alpha t}{1-\alpha}) \exp(-st) dt \Big) \\
&= \frac{1}{1-\alpha} \Big(- f'(0) + s \int_0^\infty f'(t) \exp(-st) dt \Big) \Big(\int_0^\infty \exp(-t(s + \frac{\alpha}{1-\alpha})) dt \Big) \\
&= \Big(- f'(0) - sf(0) + s^2 \int_0^\infty f(t) \exp(-st) dt \Big) \Big(\frac{1}{s + \alpha(1-s)} \Big) \\
&= \frac{s^2\mathcal{L}[f(t)] - sf(0) - f'(0)}{s + \alpha(1-s)}.
\end{aligned}$$

Finally

$$\begin{aligned}
\mathcal{L}[{}^{CF}D_t^{(\alpha+n)} f(t)](s) &= \frac{1}{1-\alpha} \mathcal{L}[f^{(n+1)}(t)] \mathcal{L}[\exp -\frac{\alpha t}{1-\alpha}] \\
&= \frac{s^{n+1}\mathcal{L}[f(t)] - s^n f(0) - s^{n-1} f'(0) - \ldots - f^{(n)}(0)}{s + \alpha(1-s)}.
\end{aligned}$$

□

The Inverse Laplace Transform

Definition 2.6. [18]

If $G(s) = \mathcal{L}[g(t)](s)$, then the inverse transform of $G(s)$, is defined as:

$$\mathcal{L}^{-1}G(s) = g(t).$$

properties of the inverse Laplace transform

- $\mathcal{L}^{-1}[a\ G_1(s) + b\ G_2(s)] = a\ g_1(t) + b\ g_2(t)$.
- $\mathcal{L}^{-1}G(s-a) = e^{at}g(t)$.
- $\mathcal{L}^{-1}\left[\frac{G(s)}{s}\right] = \int_0^t g(t)dt$.

Images of basic elementary functions

F(s)	$F^{-1}(s)$	F(s)	$F^{-1}(s)$
1	$\frac{1}{s}$	$\exp(\alpha t)\cos\beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	$\exp(\alpha t)\sin\beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$
$\exp(\alpha t)$	$\frac{1}{s-\alpha}$	$\frac{t^n}{n!}\exp(\alpha t)$	$\frac{1}{(s-\alpha)^{n+1}}$
$t\cos\beta t$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$	$t\sin\beta t$	$\frac{2s\beta}{(s^2+\beta^2)^2}$
$\cosh(\beta t)$	$\frac{s}{s^2-\beta^2}$	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$

2.2.2 The fractional integral associated to the CFFD

After the notion of fractional derivative of order $0 < \alpha < 1$, that of fractional integral of order $0 < \alpha < 1$ becomes a natural requirement. In this section we obtain the fractional integral associated to the Caputo-Fabrizio fractional derivative previously introduced.

Proposition 2.4. [10]

Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by:

$$^{CF}I^{(\alpha)}f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds,\ t \geq 0. \tag{2.6}$$

Proof.

Let $0 < \alpha < 1$. Consider now the following fractional differential equation,

$$^{CF}D^{(\alpha)}f(t) = u(t), \quad t \geq 0 \tag{2.7}$$

using Laplace transform, we obtain:

$$\mathcal{L}[^{CF}D^{(\alpha)}f(t)](s) = U(s), \quad s > 0$$

That is, using (2.5), we have that

$$\frac{(2-\alpha)M(\alpha)}{2(s+\alpha(1-s))}(sF(s) - f(0)) = U(s), \quad s > 0$$

or equivalently,

$$sF(s) - f(0) = \frac{2(s+\alpha(1-s))}{(2-\alpha)M(\alpha)}U(s), \quad s > 0$$

$$sF(s) = f(0) + \frac{2(s+\alpha(1-s))}{(2-\alpha)M(\alpha)}U(s), \quad s > 0$$

$$F(s) = \frac{1}{s}f(0) + \frac{2\alpha}{s(2-\alpha)M(\alpha)}U(s) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}U(s), \quad s > 0$$

Hence, using now well known properties of inverse Laplace transform, we that

$$\mathcal{L}^{-1}\Big[F(s)\Big] = \mathcal{L}^{-1}\Big[\frac{1}{s}(f(0))\Big] + \frac{2\alpha}{s(2-\alpha)M(\alpha)}\mathcal{L}^{-1}\Big[\frac{U(s)}{s}\Big] + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{L}^{-1}\Big[U(s)\Big]$$

From the properties of the inverse Laplace transform, we get the following expressions

$$f(0)\mathcal{L}^{-1}\Big[\frac{1}{s}\Big] = f(0)\ ,\ \mathcal{L}^{-1}\Big[\frac{U(s)}{s}\Big] = \int_0^t u(t)dt\ ,\ \mathcal{L}^{-1}\Big[U(s)\Big] = u(t).$$

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{s(2-\alpha)M(\alpha)}\int_0^t u(s)ds + f(0),\ t \geq 0 \tag{2.8}$$

In other words, the function defined as

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds + c,\ t \geq 0 \tag{2.9}$$

where $c \in \mathbb{R}$ is a constant, is also a solution of (2.7).

we can also rewrite fractional differential equation (2.7) as

$$\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t \exp\Big(-\frac{\alpha}{1-\alpha}(t-s)\Big)f'(s)ds = u(t),\ t \geq 0, \tag{2.10}$$

Or equivalently,

$$\int_0^t \exp\Big(-\frac{\alpha}{1-\alpha}(t-s)\Big)f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t)$$

$$\int_0^t \exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\exp\Big(\frac{\alpha s}{1-\alpha}\Big)f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t)$$

$$\exp\Big(-\frac{\alpha t}{1-\alpha}\Big)\int_0^t \exp\Big(\frac{\alpha s}{1-\alpha}\Big)f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t)$$

$$\int_0^t \exp\Big(\frac{\alpha}{1-\alpha}s\Big)f'(s)ds = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\exp\Big(\frac{\alpha}{1-\alpha}t\Big)u(t), \quad t \geq 0$$

Differentiating both sides of the latter equation, we obtain that,

$$\exp\Big(\frac{\alpha t}{1-\alpha}\Big)f'(t) = \exp\Big(\frac{\alpha t}{1-\alpha}\Big)\Big(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big)\Big[\frac{\alpha}{1-\alpha}u(t) + u'(t)\Big],$$

$$f'(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\Big(\frac{\alpha}{1-\alpha}u(t) + u'(t)\Big), t \geq 0.$$

Hence, integrating now from 0 to t, we deduce as in (2.8), that

$$\int_0^t f'(t)dt = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \int_0^t \left[u'(s) + \frac{\alpha}{1-\alpha}u(s)\right] ds$$

$$[f(t) - f(0)] = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(t) - u(0)] + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s)ds$$

$$f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}[u(t) - u(0)] + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(s)ds + f(0),\ t \geq 0. \tag{2.11}$$

□

2.2.3 Composition of CFFD Operators

Here we present a theoretical property related to the Caputo-Fabrizio fractional derivative.

Theorem 2.4. [11]
Let be $n \in \mathbb{N} - \{0\}, a, b \in \mathbb{R}(a < b)$ and $u \in C^n([a, b])$. Then the equality

$$\frac{d^n}{dt^n}({}^{CF}D_{at}^{(\alpha)}u(t)) = \sum_{i=1}^{n}(-1)^{n-i}\frac{\alpha^{n-i}}{(1-\alpha)^{n+1-i}}u^{(i)}(t) + (-1)^n\left(\frac{\alpha}{1-\alpha}\right)^n {}^{CF}D_{at}^{(\alpha)}u(t).$$

is true.

Proof. see [11] □

Corollary 2.1. Let be $a, b \in \mathbb{R}(a < b)$ and $C^1([a, b])$. Then the equality

$$\int_a^b ({}^{CF}D_{at}^{(\alpha)}u(t))dt = \frac{1}{\alpha}(u(b) - u(a)) - \frac{1-\alpha}{\alpha}({}^{CF}D_{ab}^{(\alpha)}u(b)).$$

is true.

In this section, we give some theoretical properties concerning the composition of Caputo-Fabrizio fractional operators.

Theorem 2.5. [11]
Let be $a, \alpha, \beta \in \mathbb{R}$ such that $0 < \alpha, \beta < 1(\alpha \neq \beta)$. Then the equality

$${}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = \frac{1}{\beta-\alpha}\left(\beta \cdot^{CF} D_{at}^{(\beta)}u(t) - \alpha \cdot^{CF} D_{at}^{(\alpha)}u(t)\right). \tag{2.12}$$

is true.

Proof. On on the first side, from the definition of Caputo-Fabrizio derivative, we deduce that

$$\begin{aligned}
{}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) &= \frac{1}{1-\alpha}\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\left({}^{CF}D_{a\tau}^{(\beta)}u(\tau)\right)' d\tau \\
&= \frac{1}{1-\alpha}\frac{1}{1-\beta}\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\Big[u'(\tau) \\
&- \int_a^\tau \frac{\beta}{1-\beta}\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)u'(s)ds\Big]d\tau \\
&= \frac{1}{1-\alpha}\frac{-\beta}{1-\beta}\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)^{CF} D_{a\tau}^{(\beta)}u(\tau)d\tau + \frac{1}{1-\beta}\,{}^{CF}D_{at}^{(\alpha)}u(t).
\end{aligned}$$

which is equivalent to

$$\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right){}^{CF}D_{a\tau}^{(\beta)}u(\tau)d\tau = -\frac{(1-\alpha)(1-\beta)}{\beta}\,{}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right)+\frac{1-\alpha}{\beta}\,{}^{CF}D_{at}^{(\alpha)}u(t). \tag{2.13}$$

On the other side, integrating by parts and considering that

$${}^{CF}D_{aa}^{(\beta)}u(a) = 0,$$

we obtain

$$\begin{aligned}
{}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) &= \frac{1}{1-\alpha}\int_a^t \exp-\frac{\alpha}{1-\alpha}(t-\tau)\left({}^{CF}D_{a\tau}^{(\beta)}u(\tau)\right)' d\tau \\
&= \frac{1}{1-\alpha}\,{}^{CF}D_{at}^{(\beta)}u(t) - \frac{\alpha}{(1-\alpha)^2}\int_a^t \exp-\frac{\alpha}{1-\alpha}(t-\tau)D_{a\tau}^{(\beta)}u(\tau)d\tau.
\end{aligned}$$

which is equivalent to

$$\int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right){}^{CF}D_{a\tau}^{(\beta)}u(\tau)d\tau = \frac{1-\alpha}{\alpha}\,{}^{CF}D_{at}^{(\beta)}u(t)-\frac{(1-\alpha)^2}{\alpha}\,{}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right). \tag{2.14}$$

Combining (2.13) with (2.14), we obtain

$$\left(\frac{(1-\alpha)(1-\beta)}{\beta}-\frac{(1-\alpha)^2}{\alpha}\right){}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = \left(-\frac{1-\alpha}{\beta}+\frac{1-\alpha}{\alpha}\right){}^{CF}D_{at}^{(\beta)}u(t).$$

$$\left(\frac{(1-\beta)}{\beta}-\frac{(1-\alpha)}{\alpha}\right){}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = -\frac{1}{\beta}\,{}^{CF}D_{at}^{(\alpha)}+\frac{1}{\alpha}\,{}^{CF}D_{at}^{(\beta)}u(t).$$

$$\left(-\frac{\beta-\alpha}{\alpha\beta}\right){}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = -\frac{1}{\beta}\,{}^{CF}D_{at}^{(\alpha)}+\frac{1}{\alpha}\,{}^{CF}D_{at}^{(\beta)}u(t).$$

$${}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = \frac{-\alpha}{\beta-\alpha}\,{}^{CF}D_{at}^{(\alpha)}u(t)+\frac{\beta}{\beta-\alpha}\,{}^{CF}D_{at}^{(\beta)}u(t).$$

$${}^{CF}D_{at}^{(\alpha)}\left({}^{CF}D_{at}^{(\beta)}u(t)\right) = \frac{1}{\beta-\alpha}\left(\beta\,.^{CF}D_{at}^{(\beta)}u(t)-\alpha\,.^{CF}D_{at}^{(\alpha)}u(t)\right).$$

□

Theorem 2.6. [11]

Let be a, $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the equality

$$ {}^{CF}D_{at}^{(\alpha)} \left({}^{CF}I_{at}^{(\alpha)} u(t) \right) = u(t) - \exp\left(-\frac{a}{1-a}(t-a) \right) u(a). \tag{2.15}$$

is true.

Proof.

$$\begin{aligned}
{}^{CF}D_{at}^{(\alpha)} \left({}^{CF}I_{a\alpha}^{(\alpha)} u(t) \right) &= \frac{1}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) \left[I_{a\tau}^{\alpha} u(\tau) \right]' d\tau \\
&= \frac{1}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) \left[(1-\alpha)u(\tau) + \alpha \int_a^{\tau} u(s)ds \right]' d\tau \\
&= \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u'(\tau)d\tau + \frac{\alpha}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u(\tau)d\tau \\
&= \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u'(\tau)d\tau + u(t) - \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u(a) \\
&- \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u'(\tau)d\tau \\
&= u(t) - \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau) \right) u(a).
\end{aligned}$$

□

Theorem 2.7. [11]

Let $a, \alpha \in \mathbb{R}$ such that $0 < \alpha < 1$. Then the equality

$$ {}^{CF}I_{at}^{(\alpha)} \left({}^{CF}D_{at}^{(\alpha)} u(t) \right) = u(t) - u(a). $$

is true.

Proof. On the one side, using definition of Caputo-Fabrizio integral, we obtain

$$ {}^{CF}I_{at}^{(\alpha)} \left({}^{CF}D_{at}^{(\alpha)} u(t) \right) = (1-\alpha) {}^{CF}D_{at}^{(\alpha)} u(t) + \alpha \int_a^t {}^{CF}D_{as}^{(\alpha)} u(s)ds. \tag{2.16}$$

On the other side, applying Theorem (2.4), we obtain

$$ \frac{d}{ds} \left[{}^{CF}D_{as}^{(\alpha)} u(s) \right] = -\frac{\alpha}{1-\alpha} {}^{CF}D_{as}^{(\alpha)} u(s) + \frac{1}{1-\alpha} u'(s), \tag{2.17}$$

Integrating (2.17) respect to s over (a,t) to t and considering that ${}^{CF}D_{aa}^{(\alpha)} u(a) = 0$, we obtain

$$ {}^{CF}D_{at}^{\alpha} u(t) = -\frac{\alpha}{1-\alpha} \int_a^t {}^{CF}D_{at}^{\alpha} u(s)ds + \frac{1}{1-\alpha} \int_a^t u'(s)ds $$

$$ {}^{CF}D_{at}^{\alpha} u(t) - \frac{1}{1-\alpha} [u(t) - u(a)] = -\frac{\alpha}{1-\alpha} \int_a^t {}^{CF}D_{at}^{\alpha} u(s)ds $$

$$\int_a^t {}^{CF}D_{as}^{(\alpha)}u(s)ds = \frac{1}{\alpha}[u(t)-u(a)] - \frac{1-\alpha}{\alpha}.{}^{CF}D_{at}^{(\alpha)}u(t). \tag{2.18}$$

Inserting the right hand side of (2.18) into (2.16), we obtain

$$\begin{aligned}{}^{CF}I^{(\alpha)}\left({}^{CF}D_{at}^{\alpha}u(t)\right) &= (1-\alpha)^{CF}D_{at}^{\alpha}u(t) + [u(t)-u(a)] - (1-\alpha)^{CF}D_{at}^{\alpha}u(t) \\ &= u(t)-u(a)\end{aligned}$$

□

Theorem 2.8. [11]

Let $0<\alpha<1, a\in\mathbb{R}$. Then the equality

$${}^{CF}I_{at}^{(\alpha)}\left(I_{at}^{\alpha}u(t)\right) = (1-\alpha)I_{at}^{(\alpha)}u(t) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}I_{at}^{(\alpha+1)}u(t).$$

is true.

Proof.

$$\begin{aligned}{}^{CF}I_{at}^{(\alpha)}\left(I_{at}^{\alpha}u(t)\right) &= (1-\alpha)I_{at}^{\alpha}u(t) + \alpha\int_a^t \; I_{a\tau}^{\alpha}u(\tau)d\tau \\ &= (1-\alpha)I_{at}^{\alpha}u(t) + \alpha\int_a^t \left[\frac{1}{\Gamma(\alpha)}\int_a^{\tau}(\tau-\xi)^{\alpha-1}u(\xi)d\xi\right]d\tau \\ &= (1-\alpha)I_{at}^{\alpha}u(t) + \frac{\alpha}{\Gamma(\alpha)}\int_a^t u(\xi)d\xi\int_{\xi}^t(\tau-\xi)^{\alpha-1}d\tau \\ &= (1-\alpha)I_{al}^{\alpha}u(t) + \frac{\alpha}{\Gamma(\alpha)}\int_a^t\frac{1}{\alpha}(t-\xi)^{\alpha}u(\xi)d\xi \\ &= (1-\alpha)I_{at}^{\alpha}u(t) + \frac{1}{\Gamma(\alpha)}\int_a^t(t-\xi)^{(\alpha+1)-1}u(\xi)d\xi \\ &= (1-\alpha)I_{au}^{\alpha}u(t) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_a^t(t-\xi)^{(\alpha+1)-1}u(\xi)d\xi \\ &= (1-\alpha)I_{at}^{\alpha}u(t) + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}I_{at}^{(\alpha+1)}u(t).\end{aligned}$$

□

Chaptre 3

Some Ordinary linear Caputo-Fabrizio fractional differential equations

3.1 Some Results of Existence and Uniqueness of the Solution

Lemma 3.1. [10]
Let $0 < \alpha < 1$. and f be a solution of the following fractional differential equation,

$$ {}^{CF}D^{(\alpha)}f(t) = 0, \quad t \geq 0. \tag{3.1}$$

Then, f is a constant function. The converse, as indicated in the introduction, is also true.

Proof. From (2.11), we obtain that the solution of (3.1) must satisfy $f(t) = f(0)$ for all $t \geq 0$. Hence, it is clear that f must be a constant function. □

Proposition 3.1. [10]
Let $0 < \alpha < 1$. Then, the unique solution of the following initial value problem

$$\begin{cases} {}^{CF}D^{(\alpha)}f(t) = \sigma(t), & t \geq 0 \\ f(0) = f_0 \in \mathbb{R} \end{cases} \tag{3.2}$$

is given by :

$$f(t) = f_0 + a_\alpha(\sigma(t) - \sigma(0)) + b_\alpha \int_0^t \sigma(t), \quad t \geq 0 \tag{3.3}$$

where

$$a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}, \; b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)}. \tag{3.4}$$

Proof. The existence of solution was previously demonstrated in Proposition (2.4).Hence, suppose that the initial value problem (3.2) have two solutions, f_1 and f_2. In that case, we have that

$$^{CF}D^{(\alpha)}f_1(t) -^{CF} D^{(\alpha)}f_2(t) = [^{CF}D^{(\alpha)}f_1 - f_2](t) = 0 \;\; and \;\; (f_1 - f_2)(0) = 0.$$

So, by Lemma (3.1), we have that $f_1 - f_2 = 0$. That is $f_1(t) = f_2(t)$ for all $t \geq 0$.
By (2.11), it is clear that the function defined by (3.3) is a solution of the fractional differential equation (3.2). Moreover, if we substitute t by 0 in (3.3), we obtain f_0.
Hence, the function defined by (3.3) is the unique solution of initial value problem (3.2). □

Now, we consider the following linear fractional differential equation

$$\begin{cases} ^{CF}D^{(\alpha)}f(t) = \lambda f(t) + u(t), & t \geq 0, \\ f(0) = f_0, \end{cases} \tag{3.5}$$

where $\lambda \in \mathbb{R}, \lambda \neq 0$ ($\lambda = 0$ corresponds to the case previously studied).
From Proposition (3.2), we have that solving equation (3.5) is equivalent to find a function f such that

$$f(t) = f_0 + a_\alpha[\lambda(f(t) - f_0 + u(t) - u(0)] + b_\alpha \int_0^t [\lambda f + u](s)ds, t \geq 0$$

where a_α, b_α are given by (3.4). Equivalently, we must find f such that

$$(1 - \lambda a_\alpha)f(t) - \lambda b_\alpha \int_0^t f(t) = (1 - \lambda a_\alpha)f_0 + a_\alpha(u(t) - u(0)) + b_\alpha \int_0^t u(t), t \geq 0.$$

If $\lambda a_\alpha = 1$, we obtain:

$$-\lambda b_\alpha \int_0^t f(s)ds = a_\alpha\Big(u(t) - u(0)\Big) + b_\alpha \int_0^t u(s)ds,$$

$$\int_0^t f(s)ds = -\frac{a_\alpha}{\lambda b_\alpha}\Big(u(t) - u(0)\Big) - \frac{b_\alpha}{\lambda b_\alpha} \int_0^t u(s)ds,$$

$$f(t) = -\frac{a_\alpha}{\lambda b_\alpha}u'(t) - \frac{1}{\lambda}u(t), t \geq 0.$$

In the other case, i.e. $\lambda a_\alpha \neq 1$, we have that:

$$f(t) - \frac{\lambda b_\alpha}{1 - \lambda a_\alpha} \int_0^t f(t) = \Phi(t), t \geq 0 \tag{3.6}$$

where

$$\Phi(t) = f_0 + \frac{a_\alpha}{1-\lambda a_\alpha}(u(t)-u(0)) + \frac{b_\alpha}{1-\lambda a_\alpha}\int_0^t u(t),\ t \geq 0.$$

The case $\lambda = 0$ is trivial, and we obtain

$$f(t) = f_0 + a_\alpha\Big(u(t) - u(0)\Big) + b_\alpha \int_0^t u(s)ds.$$

If $\lambda \neq 0$, we see that (3.6) can be rewritten as

$$f(t) - \theta \int_0^t f(t) = \Phi(t),\ t \geq 0$$

where

$$\theta = \frac{\lambda b_\alpha}{1-\lambda a_\alpha}.$$

Hence,

$$f'(t) = \theta f(t) + \Phi'(t),\ t \geq 0.$$

Thus, we have obtained an ordinary differential equation, which has a unique solution if we consider an initial condition.

Theorem 3.1. [19]

For $\sigma = \alpha + 1$, $\alpha \in (0,1)$, and $g : [0,\infty) \to \mathbb{R}$ with $g \in L_1(0,\infty)$, the following boundary value problem of Caputo-Fabrizio fractional differential equation

$$\begin{cases} {}^{CF}D^{(\sigma)} f(x) = g(x), \quad x \geqslant 0, \\ f(0) = f_0 \quad f(1) = f_1, \end{cases} \tag{3.7}$$

has the unique solution given by :

$$f(x) = f_0 + (f_1 - f_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt$$
$$+ \alpha(x-1)\int_0^x tg(t)dt - (1-\alpha)x\int_x^1 g(t)dt - \alpha x \int_x^1 (1-t)g(t)dt.$$

Proof. Applying the Laplace operator to the equation (3.7), we get

$$\mathcal{L}[{}^{CF}D^{(\sigma)} f(x)](s) = \mathcal{L}[g(x)](s).$$

Appealing the Lemma (2.1), we are led to

$$\frac{s^2F(s) - sf(0) - f'(0)}{s + \alpha(1-s)} = G(s)$$

where $F(s) = \mathcal{L}[f(x)](s)$ and $G(s) = \mathcal{L}[g(x)](s)$. Equivalently, we can rewrite the last equation as

$$F(s) = \frac{1}{s}f(0) + \frac{1}{s^2}f'(0) + \frac{1-\alpha}{s}G(s) + \frac{\alpha}{s^2}G(s).$$

The inverse Laplace operator is applied to above equation to arrive at

$$f(x) = f(0) + xf'(0) + (1-\alpha)\int_0^x g(t)dt + \alpha\int_0^x (x-t)g(t)dt. \tag{3.8}$$

Taking into account the boundary conditions, we have the desired result

$f_0 = f(0)$ and

$$f_1 = f(1) = f_0 + f'(0) + (1-\alpha)\int_0^1 g(t)dt + \alpha\int_0^1 (1-t)g(t)dt \tag{3.9}$$

or equivalently

$$f_1 - f_0 = f'(0) + (1-\alpha)\int_0^1 g(t)dt + \alpha\int_0^1 (1-t)g(t)dt,$$

by multiplying both sides by x

$$(f_1 - f_0)x = xf'(0) + (1-\alpha)x\int_0^1 g(t)dt + \alpha x\int_0^1 (1-t)g(t)dt$$

$$xf'(0) = (f_1 - f_0)x - (1-\alpha)x\int_0^1 g(t)dt - \alpha x\int_0^1 (1-t)g(t)dt,$$

substitute $xf'(0)$ into (3.8)

$$f(x) = f_0 + (f_1 - f_0)x - (1-\alpha)x\int_0^1 g(t)dt - \alpha x\int_0^1 (1-t)g(t)dt + (1-\alpha)\int_0^x g(t)dt + \alpha\int_0^x (x-t)g(t)dt$$

$$f(x) = f_0 + (f_1 - f_0)x - (1-\alpha)x\int_0^x g(t)dt - (1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_0^x (1-t)g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt$$

$$+(1-\alpha)\int_0^x g(t)dt + \alpha\int_0^x (x-t)g(t)dt$$

$$f(x) = f_0 + (f_1 - f_0)x - (1-\alpha)x\int_0^x g(t)dt - (1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_0^x g(t)dt + \alpha x\int_0^x tg(t)dt$$

$$-\alpha x\int_x^1 (1-t)g(t)dt + (1-\alpha)\int_0^x g(t)dt + \alpha x\int_0^x g(t)dt - \alpha\int_0^x tg(t)dt$$

$$f(x) = f(0) + (f_1 - f_0)x + (1-\alpha)(1-x)\int_0^x g(t)dt + \alpha(x-1)\int_0^x tg(t)dt$$

$$-(1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt.$$

For the uniqueness, as usual, we suppose that there are two solutions of the problem, say v_1 and v_2. Then we must have

$${}^{CF}D^{(\sigma)}(v_1)(x) - {}^{CF}D^{(\sigma)}(v_2)(x) = {}^{CF}D^{(\sigma)}(v_1 - v_2)(x) = {}^{CF}D^{(\alpha)}(Dv_1 - Dv_2)(x) = 0.$$

Thus, by Lemma (3.1) we get

$$Dv_1(x) = Dv_2(x).$$

This implies that $v_1(x) = v_2(x) + c$ for some constant c. But the condition $v_1(0) = v_2(0)$ leads to c = 0. That is $v_1(x) = v_2(x)$ for all $x \geqslant 0$. □

Remark 3.1. In Theorem (3.1), if we let $h(x) = g(x) - g(0)$, then $h(0) = 0$, so that the initial value problem

$$\begin{cases} {}^{CF}D^{(\sigma)}f(x) = h(x), \;\; x \geqslant 0 \\ f(0) = A, \;\; f'(0) = B \end{cases}$$

has the unique solution of much simpler form given by

$$f(x) = A + Bx + (1-\alpha)\int_0^x h(t)dt + \alpha\int_0^x (x-t)h(t)dt.$$

Theorem 3.2. [19]

If $\sigma \in (1,2)$ and $g \in L^1(0,\infty) \cap C^1[0,\infty)$, then the following linear boundary value problem of Caputo-Fabrizio fractional differential equation has the unique solution for all $\eta \in \mathbb{R}$.

$$\begin{cases} {}^{CF}D^{(\sigma)}f(x) = \eta f(x) + g(x), \quad \eta \neq 0, \quad x \geqslant 0 \\ f(0) = f_0, \quad f(1) = f_1 \end{cases} \tag{3.10}$$

Proof. The case when $\eta = 0$ is already was proved in Theorem (3.1). So, assume that $\eta \neq 0$. we see that from Theorem (3.1), the solution to (3.10) can be written as

$$f(x) = f_0 + (f_0 - f_1)x + (1-\alpha)(1-x)\int_0^x (\eta f(t) - g(t))dt + \alpha(x-1)\int_0^x t(\eta f(t) - g(t))dt$$
$$-(1-\alpha)x\int_x^1 t(\eta f(t) - g(t))dt - \alpha x\int_x^1 (1-t)(\eta f(t) - g(t))dt.$$

After simplification, we find

$$f(x) + \eta x\int_0^1 (1-\alpha t)f(t)dt - \eta\int_0^x (1-\alpha+x\alpha-t\alpha)f(t)dt = f_0 + (f_0 - f_1)x$$
$$+(1-\alpha)(1-x)\int_0^x g(t)dt + \alpha(1-x)\int_0^x tg(t)dt$$
$$-(1-\alpha)x\int_x^1 g(t)dt - \alpha x\int_x^1 (1-t)g(t)dt. \tag{3.11}$$

Differentiating the equation (3.11) twice, we have that

$$f''(x) - (1-\alpha)\eta f'(t) = (1-\alpha)g'(x) + \alpha g(x). \tag{3.12}$$

Now we have two cases to analyze. First, we assume that $(1-\alpha)\eta = 0 \Leftrightarrow \alpha = 1$, since $\eta \neq 0$. In this case, the equation (3.12) becomes

$$f''(x) = g(x).$$

This is just a second order ordinary differential equation with solution given by

$$f(x) = -f_0 x + f_0 + f_1 x + (x-1)\int_1^0 \Big(\int_1^s g(y)dy\Big)ds + \int_1^x \Big(\int_1^s g(y)dy\Big)ds.$$

The second case when $(1-\alpha)\eta \neq 0$, we have

$$f(x) = f_0 + \int_0^x e^{(1-\alpha)\eta t}\int_0^t e^{(1-\alpha)\eta s}\Big((1-\alpha)g'(s) + g(s)\Big)\,dsdt$$
$$+\frac{f_1 - f_0 - \displaystyle\int_0^1 e^{(1-\alpha)\eta t}\int_0^t e^{(1-\alpha)\eta s}\Big((1-\alpha)g'(s) + \alpha g(s)\Big)\,dsdt}{\displaystyle\int_0^1 e^{(1-\alpha)\eta t}dt}\int_0^x e^{(1-\alpha)\eta t}dt.$$

□

3.1.1 Examples

Exemple 3.1. [19]

Consider the initial value problem

$$\begin{cases} {}^{CF}D^{(\sigma)}f(x) + f(x) = 0, \\ f(0) = 1, f'(0) = 0 \end{cases}$$

where $\sigma = \alpha + 1$ with $\alpha \in (0,1)$.

Applying the Laplace transformation leads to have

$$F(s)(s^2 + s + \alpha(1-s)) = s.$$

Now, the inverse Laplace transformation gives the exact solution as follows

$f(x) = \exp(x(\alpha/2 - 1/2))(\cosh(x(\alpha^2/4 - 3\alpha/2 + 1/4)^{1/2}) + \sinh(x(\alpha^2/4$
$- 3\alpha/2 + 1/4)^{1/2})(\alpha/2 - 1/2))/(\alpha^2/4 - 3\alpha/2 + 1/4)^{1/2})$.

Exemple 3.2. Consider the system of fractional algebraic-differential equations

$$\begin{cases} {}^{CF}D^{(1/2)}u(x) - x\sin(x) + u(t) - (1+x)\sin(x) = 0 \\ u(0) = 1. \end{cases}$$

Applying the Laplace transformation, with $v(x) = \sin x$ one gets

$$\frac{sU(s) - 1}{(s+1)/2} + V(s) + sV'(s) + U(s) - V'(s) = 0$$
$$V(s)\frac{1}{s^2}, V'(s) = -\frac{2s}{(s^2+1)^2},$$
$$U(s) = \frac{s(s+1)}{(1+s^2)^2} + \frac{1}{s+1}.$$

Now, the inverse Laplace transform gives the exact solution

$$u(x) = \frac{x+1}{2}\sin x + \frac{x}{2}\cos x + \exp(-x).$$

Exemple 3.3. Consider the boundary value problem

$$\begin{cases} {}^{CF}D^{(3/2)}u(x) = \lambda u(x), \\ u(0) = 0, u(1) = 1. \end{cases}$$

This is the equation given in the problem (3.10) with $\sigma = 1 + 1/2$ and $u_0 = 0, u_1 = 1$. Thus, the exact solution given by

$$u(x) = \frac{1}{\int_0^1 e^{(\lambda/2)t}dt}\int_0^x e^{(\lambda/2)t}dt = \frac{e^{\lambda x/2}-1}{e^{\lambda/2}-1}.$$

Chaptre 4

Some Ordinary Nonlinear Caputo-Fabrizio fractional differential equations

4.1 Results of applying the Banach's Principle of Contraction theorem to boundary value problems

We prove the existence and uniqness of the nonlinear boundary value problems of the Caputo-Fabrizio differential equations by the help of the Banach contraction principle. [19] Let $\mathscr{C}(I)$ be the Banach space of continuous functions on $I = [0,1]$ with supremum norm

$$\|x\| = \sup_{s\in[0,1]} |x(s)|, \;\; x \in \mathscr{C}(I).$$

We now state the existence and uniqness of the solution in the next theorem.

Theorem 4.1. [19]
If $\sigma = \alpha + 1, \;\; \alpha \in (0.1]$ and $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the property that

$$| \; F(x,u_1) - F(x,u_2) \; | \leq q|u_1 - u_2| \;\;\; u_1, u_2 \in \mathbb{R}, \quad q > 0.$$

then the boundary value problem

$$\begin{cases} {}^{CF}D^{(\sigma)}u(x) = F(x,u(x)), \quad x \geqslant 0, \\ u(0) = u_0, \quad u(1) = u_1, \end{cases} \tag{4.1}$$

has a unique solution in $\mathscr{C}(I)$ provided $q < 1$.

Proof.

Let the operator $N : \mathscr{C}(I) \to \mathscr{C}(I)$ be is given by the solution expression in theorem (3.1),

$$(Nu)(x) = u_0 + (u_1 - u_0)x + (1-\alpha)(1-x)\int_0^x F(t,u(t))dt + \alpha(x-1)\int_0^x tF(t,u(t))dt$$
$$-(1-\alpha)x\int_x^1 F(t,u(t))dt - \alpha x\int_x^1 (1-t)F(t,u(t))dt.$$

We see that the solution for the problem (4.1) is the fixed point of the map N.

For $u, v \in \mathscr{C}(I)$ and $0 \leqslant t \leqslant 1$, we find that

$$\begin{aligned}\Big|(Nu)(x) - (Nv)(x)\Big| = &\Big|(1-\alpha)(1-x)\int_0^x (F(t,u(t)) - F(t,v(t)))dt \\ &+ \alpha(1-x)\int_0^x t(F(t,u(t)) - F(t,v(t)))dt \\ &-(1-\alpha)x\int_x^1 (F(t,u(t)) - F(t,v(t))dt - \alpha x\int_x^1 (1-t)(F(t,u(t)) - F(t,v(t))dt\Big| \\ &\le (1-\alpha)(1-x)xq \parallel u - v \parallel \\ &+ \alpha(1-x)\frac{x^2}{2}q \parallel u-v \parallel + (1-\alpha)x(1-x)q \parallel u-v \parallel + \alpha x\frac{(1-x)^2}{2}q \parallel u-v \parallel \\ &= (1-x)x\frac{4-3\alpha}{2}q \parallel u-v \parallel \le \sup_{x\in[0,1]}(1-x)x\frac{4-3\alpha}{2}q \parallel u-v \parallel \\ &\le \frac{4-3\alpha}{8}q \parallel u-v \parallel \le q \parallel u-v \parallel .\end{aligned}$$

Since $q < 1$, the operator N is a contraction, and by the Banach contraction theorem N must have a unique fixed point that is the solution of the problem (4.1). □

4.2 Results of applying the Banach's Principle of Contraction theorem to initial value problems

Theorem 4.2. [10]

Let $0 < \alpha < 1, T > 0$ and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ a continuous function such that there exits $L > 0$ satisfying

$$|F(x,t_1) - F(x,t_2)| \le L|t_1 - t_2| \quad \forall\ t_1, t_2 \in \mathbb{R}.$$

If $(a_\alpha + b_\alpha T)L < 1$, then the initial value problem

$$\begin{cases} {}^{CF}D^{(\alpha)}u(x) = F(x,u(x)), & x \in [0,T] \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{4.2}$$

has a unique solution on $\mathscr{C}[0,T]$.

Proof. Let $\mathscr{C}[0,T]$ be the space of all continuous functions defined on the interval $[0,T]$ endowed with the usual supremum norm, that is,

$$\|u\| = \sup_{x\in[0,T]} |u| \quad \forall u \in \mathscr{C}[0,T].$$

We consider the operator $N : \mathscr{C}[0,T] \to \mathscr{C}[0,T]$, by proposition (3.1) the solution of (4.2) is given as follows

$$Nu(x) = c + a_\alpha F(x,u(x)) + b_\alpha \int_0^x F(t,u(t))dt \quad \forall u \in \mathscr{C}[0,T],$$

where

$$c = -a_\alpha F(0,u_0) + u_0.$$

Finding a solution of (4.2) in $\mathscr{C}[0,T]$ is equivalent to finding a fixed point of the operator N . Since for all $u_1, u_2 \in \mathscr{C}[0,T]$ and all $x \in [0,T]$ we have that

$$\begin{aligned}
|Nu_1(x) - Nu_2(x)| &= \left| a_\alpha \left(F(x,u_1(x)) - F(x,u_2(x))\right) + b_\alpha (\int_0^x F(t,u_1(t))dt - \int_0^x F(t,u_2(t))dt) \right| \\
&\le a_\alpha |F(x,u_1(x)) - F(x,u_2(x))| + b_\alpha \int_0^x |F(t,u_1(t)) - F(t,u_2(t))|ds \\
&\le a_\alpha L|u_1(x) - u_2(x)| + b_\alpha L \int_0^x |u_1(t) - u_2(t)|dt \\
&\le (a_\alpha + b_\alpha T)L \parallel u_1 - u_2 \parallel .
\end{aligned}$$

We conclude that operator N is a contraction. The statement follows now from Banach's Fixed Point Theorem. □

4.3 Results of applying the Krasnoselskii's fixed point theorem to Caputo-Fabrizio fractional differential equations

In this section, we use Krasnoselskii's fixed point theorem to obtain some results for the existence and uniqueness of a solution to caputo-fabrizio fractional differential equations. Hence, the equation reads as [7]

$$\begin{cases} {}_0^{CF}D_x^{(\alpha)} u(x) = f\left(x, u(x), {}_0^{CF}D_x^{(\alpha)} u(x)\right), & x \in [0,T] = I \\ u(0) = u_0, \quad u_0 \in \mathbb{R} \end{cases} \tag{4.3}$$

Lemma 4.1. [7]

Let $v \in \mathscr{C}[0,T]$, then the solution of fractional differential equations

$$\begin{cases} {}_0^{CF}D_x^{(\alpha)}u(x) = v(x), & x \in [0,T], 0 < \alpha \leq 1, \\ u(0) = u_0, & u_0 \in \mathbb{R} \end{cases} \tag{4.4}$$

is given as

$$u(x) = u_0 + B_\alpha [v(x) - v_0] + \bar{B}_\alpha \int_0^t v(\tau) d\tau \tag{4.5}$$

where

$$B_\alpha = \frac{(1-\alpha)}{M(\alpha)}, \bar{B}_\alpha = \frac{\alpha}{M(\alpha)}.$$

Proof. Using the definition of ${}_0^{CF}I_x^{(\alpha)}$, (4.4) implies that

$$u(x) = c + B_\alpha v(x) + \bar{B}_\alpha \int_0^x v(\tau) d\tau, \quad c \in \mathbb{R} \tag{4.6}$$

Using the initial condition $u(0) = u_0$ and $v(0) = v_0 \in \mathbb{R}$, from (4.6), we get $c = u_0 - B_\alpha v_0$. Hence by plugging the value of c in (4.6), we get (4.5).

□

Remark 4.1. Henceforth, for simplicity, we use ${}_0^{CF}D_x^{(\alpha)}u(x) = g_u(x)$ for the implicit term in our analysis. Further, for simplicity, we use $f(0, u(0), g_{\bar{u}}(0)) = f_0$.

Lemma 4.2. [7]

Under the conditions of Lemma (4.1), the solution of (4.3) is given by

$$u(x) = u_0 + B_{(\alpha)} [f(x, u(x), g_u(x)) - f_0] + \bar{B}_{(\alpha)} \int_0^t f(\tau, u(\tau), g_u(\tau)) d\tau. \tag{4.7}$$

To proceed further, we assume that

(H_1)There exist $L_f > 0$ and $0 < M_f < 1$ such that

$$|f(x, u, g_u) - f(x, \bar{u}, g_{\bar{u}})| \leq L_f |u - \bar{u}| + M_f |g_u - g_{\bar{u}}| \ \forall\ u, \bar{u}, g_u, g_{\bar{u}} \in \mathbb{R}.$$

Let $X = \mathscr{C}(I)$ be a Banach space with norm $\|x\| = \sup_{x \in I} |u(x)|$.

Theorem 4.3. [7]

Under the assumption (H_1), if the condition $\left(B_\alpha + \bar{B}_\alpha T\right) \frac{L_f}{1-M_f} < 1$ holds, then the considered problem (4.3) has a unique solution.

Proof. Define an operator $N : X \to X$ by using (4.7) as

$$Nu(x) = u_0 + B_\alpha [f(x, u(x), g_u(x)) - f_0] + \bar{B}_\alpha \int_0^x f(\tau, u(\tau), g_u(\tau)) d\tau. \tag{4.8}$$

Then for any $u, \bar{u} \in X$, from (4.8), we have

$$
\begin{aligned}
\parallel Nu - N\bar{u} \parallel &= \sup_{x\in I}\Big|Nu(x) - N\bar{u}(x)\Big| \\
&= \sup_{x\in I}\Big|B_\alpha[f(x,u(x),g_u(x)) - f(x,\bar{u}(x),g_{\bar{u}}(x))] \\
&+ \bar{B}_\alpha\int_0^x [f(t,u(t),g_u(t)) - f(t,\bar{u}(t),g_{\bar{u}}(t))]dt\Big| \\
&\leq \sup_{x\in I}\Big[B_\alpha|f(x,u(x),g_u(x)) - f(x,\bar{u}(x),g_{\bar{u}}(x))| \\
&+ \bar{B}_\alpha\int_0^x |f(t,u(t),g_u(t)) - f(,\bar{u}(t),g_{\bar{u}}(t))|dt\Big] \\
&\leq B_\alpha \sup_{x\in I}\Big[L_f \mid u-\bar{u}\mid + M_f \mid g_u(x) - g_{\bar{u}}(t)\mid\Big] \\
&+ T\bar{B}_\alpha \sup_{x\in I}\Big[L_f \mid u-\bar{u}\mid + M_f \mid g_u(x) - g_{\bar{u}}(x)\mid\Big] \\
&\leq B_\alpha L_f \parallel u-\bar{u}\parallel + B_\alpha M_f \parallel g_u(x) - g_{\bar{u}}(x)\parallel \\
&+ T\bar{B}_\alpha L_f \parallel u-\bar{u}\parallel + T\bar{B}_\alpha M_f \parallel g_u(x) - g_{\bar{u}}(x)\parallel \\
&= (B_\alpha + \bar{B}_\alpha T)\Big(L_f \parallel u-\bar{u}\parallel + M_f \parallel g_u(x) - g_{\bar{u}}(x)\parallel\Big) \\
&\leq (B_\alpha + \bar{B}_\alpha T)\Big(L_f \parallel u-\bar{u}\parallel + M_f \sup_{x\in I}|f(x,u(x),g_u(x)) - f(x,\bar{u}(x),g_{\bar{u}}(x))|\Big) \\
&\leq (B_\alpha + \bar{B}_\alpha T)\Big(L_f \parallel u-\bar{u}\parallel + M_f\Big[L_f \parallel u-\bar{u}\parallel + M_f \parallel g_u(x) - g_{\bar{u}}(x)\parallel\Big]\Big)
\end{aligned}
$$

$After\ consecutive\ compensation,\ we\ find$

$$
\begin{aligned}
&\leq (B_\alpha + \bar{B}_\alpha T)L_f \parallel u-\bar{u}\parallel \Big[1 + M_f + (M_f)^2 + (M_f)^3 + \ldots + (M_f)^n\Big] \\
&\leq (B_\alpha + \bar{B}_\alpha T)L_f(\frac{1}{1-M_f}) \parallel u-\bar{u}\parallel .
\end{aligned}
$$

Hence N is a contraction, therefore N has a unique fixed point. Hence the corresponding problem (4.3) has a unique solution. □

Theorem 4.4. [7]

Let the given assumption hold:

(H_2) There exist constants $a_f, b_f, c_f > 0$ with $0 < c_f < 1$ such that

$$|f(x,u,v)| \leq a_f + b_f|u| + c_f|v|.$$

Under the assumption (H_2), if $0 < B_\alpha\frac{L_f}{1-M_f} < 1$ holds, then the considered problem (4.3) has at least one solution.

Proof. Let us define two operators from (4.7) as

$$N_1u(x) = u_0 + B_\alpha \left[f\left(x, u(x), g_u(x)\right) - f_0\right] \tag{4.9}$$

and

$$N_2u(x) = \bar{B}_\alpha \int_0^x f\left(\tau, u(\tau), g_{\bar{u}}(\tau)\right). \tag{4.10}$$

Let us define a set $E = \{u \in X : \|u\| \le r\}$. Since f is continuous, so is N_1, and letting $u, \bar{u} \in E$, from (4.9), we have

$$\begin{aligned} \|N_1u - N_1\bar{u}\| &= \sup_{x\in I} \left|B_\alpha \left(f\left(x, u(x), g_u(x)\right) - f\left(x, \bar{u}(x), g_{\bar{u}}(x)\right)\right)\right| \\ &\le \frac{B_\alpha L_f}{1 - M_f}\|u - \bar{u}\|. \end{aligned}$$

Hence N_1 is a contraction. Next to prove that N_2 is compact and continuous, for any $u \in E$, we have from (4.10)

$$\begin{aligned} \|N_2u\| = \sup_{x\in I} |N_2u(x)| &= \sup_{x\in I} \left|\bar{B}_\alpha \int_0^x f\left(\tau, u(\tau), g_u(\tau)\right) d\tau\right| \\ &\le \frac{\bar{B}_\alpha \left(a_f + b_f r\right)}{1 - c_f} = A, \end{aligned}$$

which implies that $\|N_2u\| \le A$. Thus N_2 is bounded. Next, letting $x_1 < x_2$ in I, we have

$$\begin{aligned} \left|N_2u\left(x_2\right) - N_2u\left(x_1\right)\right| &= \left|\bar{B}_\alpha \int_0^{x_2} f\left(\tau, u(\tau), h_u(\tau)\right) d\tau - \bar{B}_\alpha \int_0^{x_1} f\left(\tau, u(\tau), g_u(\tau)\right) d\tau\right| \\ &\le \bar{B}_\alpha \int_0^{x_2} \left|f\left(\tau, u(\tau), g_u(\tau)\right)\right| d\tau + \bar{B}_\alpha \int_0^{x_1} \left|f\left(\tau, u(\tau), g_u(\tau)\right)\right| d\tau \\ &\le \bar{B}_\alpha \int_0^{x_2} \frac{\left(a_f + b_f r\right)}{1 - c_f} d\tau + \bar{B}_\alpha \int_0^{x_1} \frac{\left(a_f + b_f r\right)}{1 - c_f} d\tau \end{aligned}$$

which implies that

$$\left|N_2u\left(x_2\right) - N_2u\left(x_1\right)\right| \le \bar{B}_\alpha \left(\frac{a_f + b_f r}{1 - c_f}\right)\left(x_2 - x_1\right). \tag{4.11}$$

From (4.11), we see that if $x_1 \to x_2$, then the right-hand side of (4.11) goes to zero, so $\left|N_2u\left(x_2\right) - N_2u\left(x_1\right)\right| \to 0$ as $x_1 \to x_2$. Thus the operator defined in (4.10), N_1 is continuous. Also $N_2(E) \subset E$, therefore N_2 is compact and, due to Arzelá–Ascoli theorem, N has at least one fixed point. Hence from Krasnoselskii's fixed point theorem the corresponding problem has at least one solution. □

Chaptre 5

Application of CFFD to PDE of Korteweg-de Vries-Burgers Equation

5.1 Study of the existence and uniqueness of solution

We aim to show in this section the existence and uniqueness of a solution to the (KDVB) equation with two perturbation's levels using the new definition Caputo-Fabrizio fractional derivative with no singular kernel. Hence, the equation reads as [6]

$$^{CF}D_t^{(\alpha)}u(x,t) = \lambda u_{xx} - 2uu_x - \eta u_{xxx} \tag{5.1}$$

where λ and η are the perturbation parameters, $^{CF}D_t^{(\alpha)}$ is the CFFD given in (2.10) with initial condition.

$$u(x,0) = h(x). \tag{5.2}$$

To proceed with existence results for the model (5.1) and (5.2), we exploit the expression of integral (2.6). Then,

$$u(x,t) - u(x,0) =^{CF} I_t^{(\alpha)}(\lambda u_{xx} - 2uu_x - \eta u_{xxx}).$$

Equivalently,

$$u(x,t)-u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(\lambda u_{xx}-2uu_x-\eta u_{xxx})+\frac{2\alpha)}{(2-\alpha)M(\alpha)}\int_0^t(\lambda u_{xx}-2uu_x-\eta u_{xxx})d\tau. \tag{5.3}$$

Let us now put

$$N(x,t,u,\lambda,\eta) = \lambda u_{xx} - 2uu_x - \eta u_{xxx}.$$

We have to find a positive real number k such that

$$\|N(x,t,u,\lambda,\eta) - N(x,t,v,\lambda,\eta)\| \leq k\|u-v\|.$$

In fact

$$\begin{aligned}
&N(x,t,u,\lambda,\eta) - N(x,t,v,\lambda,\eta)\\
&\qquad = (\lambda u_{xx} - 2uu_x - \eta u_{xxx}) - (\lambda v_{xx} - 2vv_x - \eta v_{xxx})\\
&\qquad = \lambda(u_{xx} - v_{xx}) - 2(vv_x - uu_x) + \eta(v_{xxx} - u_{xxx}).
\end{aligned}$$

Exploiting the norm's properties leads to

$$\begin{aligned}
&\|N(x,t,u,\lambda,\eta) - N(x,t,v,\lambda,\eta)\|\\
&\qquad = \|\lambda\left(u_{xx} - v_{xx}\right) + 2\left(vv_x - uu_x\right) + \eta\left(v_{xxx} - u_{xxx}\right)\|\\
&\qquad \leq \lambda\left\|u_{xx} - v_{xx}\right\| + 2\left\|vv_x - uu_x\right\| + \eta\left\|v_{xxx} - u_{xxx}\right\|\\
&\qquad \leq \lambda\left\|\partial_{xx}(u-v)\right\| + \left\|\partial_x\left(v^2 - u^2\right)\right\| + \eta\left\|\partial_{xxx}(v-u)\right\|.
\end{aligned}$$

Because of assumption that u and v are bounded, there is a positive constant $c > 0$ such that $\|u\| \leq c$ and $\|v\| \leq c$. Then, their first order derivative function ∂_x satisfies the Lipschitz condition and there is a number $k_1 \geq 0$ such that

$$\begin{aligned}
&\|N(x,t,u,\lambda,\eta) - N(x,t,v,\lambda,\eta)\|\\
&\qquad \leq \lambda\ k_1^2\|u-v\| + k_1\left\|v^2 - u^2\right\| + \eta\ k_1^3\|u-v\|\\
&\qquad \leq \lambda\ k_1^2\|u-v\| + k_1\|u+v\| \cdot \|u-v\| + \eta\ k_1^3\|u-v\|\\
&\qquad \leq \left[\lambda k_1^2 + 2ck_1 + \eta\ k_1^3\right]\|u-v\|,
\end{aligned}$$

where we have used the bounded condition (5.2). Hence,

$$\|N(x,t,u,\lambda,\eta) - N(x,t,v,\lambda,\eta)\| \leq k\|u-v\|$$

with $k = \lambda k_1^2 + 2ck_1 + \eta k_1^3$. This shows the Lipschitz condition for N. Now we can state the following theorem.

Proposition 5.1. [6]
Under the condition that $\dfrac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} + \dfrac{2kt\alpha}{(2-\alpha)M(\alpha)} < 1$, then the non-linear time fractional Korteweg-de Vries-Burgers model with two perturbation's levels and no singular kernel

$$\begin{cases} {}^{CF}D_t^{(\alpha)}u(x,t) = \lambda u_{xx} - 2uu_x - \eta u_{xxx}, \\ u(x,0) = h(x), \end{cases} \tag{5.4}$$

admits a unique solution that is continuous.

Proof.

To prove it, we consider (5.3)

$$u(x,t) - u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N(x,t,u,\lambda,\eta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t N(x,\tau,u,\lambda,\eta) d\tau,$$

that suggests the following recurrence formula

$$\begin{aligned} u_0(x,t) &= u(x,0), \\ u_n(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N\left(x,t,u_{n-1},\lambda,\eta\right) \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t N\left(x,\tau,u_{n-1},\lambda,\eta\right) d\tau. \end{aligned}$$

Let

$$\bar{u}(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{5.5}$$

We aim to show that $\bar{u}(x,t) = u(x,t)$ is a solution that is continuous. Let us set

$$G_n(x,t) = u_n(x,t) - u_{n-1}(x,t).$$

It is obvious that

$$u_n(x,t) = \sum_{m=0}^{n} G_m(x,t).$$

Furthermore, in a more detailed way we have

$$G_n(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left[N\left(x,t,u_{n-1},\lambda,\eta\right) - N\left(x,t,u_{n-2},\lambda,\eta\right)\right]$$

$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(N\left(x,\tau,u_{n-1},\lambda,\eta\right) - N\left(x,\tau,u_{n-2},\lambda,\eta\right)\right) d\tau.$$

Taking the norm of the later equation gives

$$\begin{aligned} \|G_n(x,t)\| &= \|u_n(x,t) - u_{n-1}(x,t)\| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left\|N\left(x,t,u_{n-1},\lambda,\eta\right) - N\left(x,t,u_{n-2},\lambda,\eta\right)\right\| \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \left\| \int_0^t \left[N\left(x,\tau,u_{n-1},\lambda,\eta\right) - N\left(x,\tau,u_{n-2},\lambda,\eta\right)\right] d\tau \right\| \\ &\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left\|N\left(x,t,u_{n-1},\lambda,\eta\right) - N\left(x,t,u_{n-2},\lambda,\eta\right)\right\| \\ &\quad + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\|N\left(x,\tau,u_{n-1},\lambda,\eta\right) - N\left(x,\tau,u_{n-2},\lambda,\eta\right)\right\| d\tau. \end{aligned}$$

Using the Lipschitz condition for N yields

$$\|G_n(x,t)\| \le \frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} \left\|u_{n-1} - u_{n-2}\right\| + \frac{2k\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left\|u_{n-1} - u_{n-2}\right\| d\tau,$$

equivalent to

$$\| G_n(x,t) \| \leq \frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} \| G_{n-1} \| + \frac{2k\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|G_{n-1}\| \, d\tau.$$

$$\| G_n(x,t) \| \leq \left[\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2k\alpha t}{(2-\alpha)M(\alpha)}\right] \| G_{n-1} \|, \tag{5.6}$$

similarly

$$\begin{aligned} \| G_{n-1}(x,t) \| &\leq \frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} \| u_{n-2} - u_{n-3} \| + \frac{2k\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|u_{n-2} - u_{n-3}\| \, d\tau. \\ &\leq \frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} \| G_{n-2} \| + \frac{2k\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|G_{n-2}\| \, d\tau \\ &\leq \left[\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2k\alpha t}{(2-\alpha)M(\alpha)}\right] \| G_{n-2} \| . \end{aligned}$$

Get back to (5.6) become

$$\begin{aligned} \| G_n(x,t) \| &\leq \left[\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2k\alpha t}{(2-\alpha)M(\alpha)}\right] \left[\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2\alpha t}{(2-\alpha)M(\alpha)}\right] \| G_{n-2} \| \\ &\leq \left(\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2k\alpha t}{(2-\alpha)M(\alpha)}\right)^2 \| G_{n-2} \| . \end{aligned}$$

The recursive's principle gives

$$\|G_n(x,t)\| \leq \left(\frac{2(1-\alpha)k}{(2-\alpha)M(\alpha)} + \frac{2k\alpha t}{(2-\alpha)M(\alpha)}\right)^n u(x,0),$$

which proves that the solution exists and is continuous. To show that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

is the solution of the model (5.4), we let

$$V_n(x,t) = \bar{u}(x,t) - u_n(x,t) \quad \text{for} \quad n \in N.$$

Hence, from (5.5), the difference $V_n(x,t)$ between $\bar{u}(x,t)$ and $u_n(x,t)$ should tend to zero as $n \to \infty$. Indeed

$$\begin{aligned} \bar{u} - u_{n-1} &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left[N(x,t,u,\lambda,\eta) - N(x,t,u_n,\lambda,\eta)\right] \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(N(x,\tau,u,\lambda,\eta) - N(x,\tau,u_n,\lambda,\eta)\right) d\tau, \end{aligned}$$

giving

$$\begin{aligned} \|\bar{u}(x,t) - u_{n+1}\| &\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \|N(x,t,u,\lambda,\eta) - N(x,t,u_n,\lambda,\eta)\| \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|N(x,\tau,u,\lambda,\eta) - N(x,\tau,u_n,\lambda,\eta)\| \, d\tau \\ &\leq \frac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} \|u - u_n\| + \frac{2kt\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|u - u_n\| \, d\tau \\ &\leq \frac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} \|V_n\| + \frac{2kt\alpha}{(2-\alpha)M(\alpha)} \int_0^t \|V_n\| \, d\tau. \end{aligned}$$

Then indeed when $n \to \infty$, then $V_n \to 0$ and the right hand side gives

$$\lim_{n\to\infty} u_n = \bar{u}.$$

We can take $u(x,t) = \bar{u}(x,t)$ as a solution of (5.4) that is continuous. Furthermore, applying the lipschitz condition for N, we have the following:

$$u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N(x,t,u,\lambda,\eta) - \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t N(x,t,u,\lambda,\eta) d\tau$$
$$= R_n(x,t) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} \left(N\left(x,\tau,u_{n-1},\lambda,\eta\right) - N(x,t,u,\lambda,\eta)\right)$$
$$+ \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t \left(N\left(x,\tau,u_{n-1},\lambda,\eta\right) - N(x,t,u,\lambda,\eta)\right) d\tau.$$

This yields

$$\|u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N(x,t,u,\lambda,\eta) - \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t N(x,t,u,\lambda,\eta) d\tau\|$$
$$= \|G_n(x,t)\| + \left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\theta t\alpha}{(2-\alpha)M(\alpha)}\right) \|G_{n-1}(x,t)\|.$$

Passing to the limit when $n \to 0$ and considering the initial condition, we have

$$u(x,t) = u(x,0) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} N(x,t,u,\lambda,\eta) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t N(x,t,u,\lambda,\eta) d\tau.$$

For uniqueness we consider u and v be two different solutions of the model (5.4) then, the Lipschitz condition for N yields

$$\|u - v\| \le \frac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} \|u - v\| + \frac{2kt\alpha}{(2-\alpha)M(\alpha)} \|u - v\|$$

rearranged to be

$$\|u - v\| \left(1 - \frac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} - \frac{2kt\alpha}{(2-\alpha)M(\alpha)}\right) \le 0.$$

Then, $\|u - v\| = 0$ if

$$1 > \frac{2k(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2kt\alpha}{(2-\alpha)M(\alpha)},$$

and the proposition is proved. □

5.2 Numerical approximations scheme

Definition 5.1. [1]

1. Let $f(t)$ be a function in $\mathscr{C}^2[a,b]$ and let the order of the fractional derivative be $0 < \alpha \le 1$, then the first- order approximation of the Caputo–Fabrizio derivative at a point t_n is

$$ {}_0^{CF}D_t^{(\alpha)}\left(f(t_n)\right) = \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} \left(\frac{f^{j+1} - f^j}{k}\right) d_{j,k} + \frac{M(\alpha)}{\alpha} \sum_{j=1}^{n} d_{j,k} O(k) \tag{5.7}$$

where

$$d_{j,k} = -\exp\left[-\alpha\frac{k}{1-\alpha}(n-j+1)\right] + \exp\left[-\alpha\frac{k}{1-\alpha}(n-j)\right].$$

2. Let $f(x,t)$ be a function in $\mathscr{C}^2([a,b]\times[0,T])$ and let the order of the fractional derivative be $0<\alpha\leq 1$, then the first-order approximation of the Caputo–Fabrizio derivative at a point (x_m,t_n) is

$$\begin{aligned}{}_0^{CF}D_x^{(\alpha)}\left(f\left(x_m,t_k\right)\right) = \frac{M(\alpha)}{1-\alpha}\sum_{l=1}^{m}\Bigg\{&\frac{\left(f_{i+1}^{k+1}-f_{i-1}^{k+1}\right)-\left(f_{i+1}^{k}-f_{i-1}^{k}\right)}{4i}\frac{(1-\alpha)\sqrt{\pi}}{2\alpha}\\ &\left\{\operatorname{erf}\left[(m-l)\frac{\alpha i}{1-\alpha}\right]-\operatorname{erf}\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\right\}\Bigg\}\\ +O(i)\frac{(1-\alpha)}{2\alpha}\sum_{l=1}^{m}&\left\{\operatorname{erf}\left[(m-l)\frac{\alpha i}{1-\alpha}\right]-\operatorname{erf}\left[(m-l+1)\frac{\alpha i}{1-\alpha}\right]\right\}.\end{aligned}$$

Theorem 5.1. [1]

Let $f(x,t)$ be twice differentiable on both x and t directions, then the second derivative approximation of the Caputo–Fabrizio derivative of $f(x,t)$ is given as

$$\begin{aligned}{}_0^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right) = \frac{1}{2}\sum_{k=1}^{j}&\left\{\frac{\left(f_{i+1}^{k+1}-2f_i^{k+1}+f_{i-1}^{k+1}\right)+\left(f_{i+1}^{k}-2f_i^{k}+f_{i-1}^{k}\right)}{2(\Delta x)^2}\right\}\\ &\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}+O\left((\Delta x)^2\right).\end{aligned}$$

Proof. The corresponding second order of the new fractional derivative is given by:

$$^{CF}{}_0D_x^{(\alpha)}(f(x,t)) = \frac{\alpha}{(1-\alpha)\sqrt{\pi}}\int_0^x\frac{\partial^2 f(y,t)}{\partial y^2}\exp\left[-\frac{\alpha^2(x-y)^2}{(1-\alpha)^2}\right]dy.$$

However, for any given x_j, we have

$${}_0^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right) = \frac{\alpha}{(1-\alpha)\sqrt{\pi}}\sum_{i=1}^{j}\int_{x_k}^{x_{k+1}}\frac{\partial^2 f(y,t)}{\partial y^2}\exp\left[-\frac{\alpha^2\left(x_j-y\right)^2}{(1-\alpha)^2}\right]dy.$$

Using the Crank–Nicolson scheme for the usual second-order derivative, the above equation can be reformulated as

$$\begin{aligned}{}_0{}^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right) = \frac{\alpha}{(1-\alpha)\sqrt{\pi}}\sum_{i=1}^{j}&\left\{\frac{\left(f_{i+1}^{k+1}-2f_i^{k+1}+f_{i-1}^{k+1}\right)+\left(f_{i+1}^{k}-2f_i^{k}+f_{i-1}^{k}\right)}{2(\Delta x)^2}+O\left((\Delta x)^2\right)\right\}\\ &\left[\int_{x_k}^{x_{k+1}}\exp\left[-\frac{\alpha^2\left(x_j-y\right)^2}{(1-\alpha)^2}\right]dy\right], f\left(x_j,t_i\right)=f_i^j.\end{aligned}\tag{5.8}$$

Nevertheless, the integral in the right-hand side is evaluated as

$$\int_{x_k}^{x_{k+1}}\exp\left[-\frac{\alpha^2\left(x_j-y\right)^2}{(1-\alpha)^2}\right]dy = \frac{(1-\alpha)\sqrt{\pi}}{2\alpha}\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}.$$

Therefore, equation (5.8) becomes

$$
{}_0^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right)=\frac{1}{2}\sum_{i=1}^{j}\left\{\frac{\left(f_{i+1}^{k+1}-2f_i^{k+1}+f_{i-1}^{k+1}\right)+\left(f_{i+1}^{k}-2f_i^{k}+f_{i-1}^{k}\right)}{2(\Delta x)^2}+O\left((\Delta x)^2\right)\right\}
$$
$$
\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}.
$$

The above equation can be rewritten as follows

$$
{}_0^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right)=\frac{1}{2}\sum_{i=1}^{j}\left\{\frac{\left(f_{i+1}^{k+1}-2f_i^{k+1}+f_{i-1}^{k+1}\right)+\left(f_{i+1}^{k}-2f_i^{k}+f_{i-1}^{k}\right)}{2(\Delta x)^2}\right\} \tag{5.9}
$$
$$
\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}
$$
$$
+\sum_{i=1}^{j}\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}O\left((\Delta x)^2\right).
$$

Note that

$$
\sum_{k=1}^{j}\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}=\operatorname{erf}\left[\frac{-j\alpha}{1-\alpha}\Delta x\right].
$$

Using the Abromowitz and Stegun series approximation of the error function, equation (5.9) is reduced to

$$
{}_0^{CF}D_x^{(\alpha)}\left(f\left(x_j,t\right)\right)=\frac{1}{2}\sum_{i=1}^{j}\left\{\frac{\left(f_{i+1}^{k+1}-2f_i^{k+1}+f_{i-1}^{k+1}\right)+\left(f_{i+1}^{k}-2f_i^{k}+f_{i-1}^{k}\right)}{2(\Delta x)^2}\right\}
$$
$$
\left\{\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_{k+1}\right)\right]-\operatorname{erf}\left[\frac{\alpha}{1-\alpha}\left(x_j-x_k\right)\right]\right\}+O\left((\Delta x)^2\right).
$$

□

Now, applying the scheme (5.7) to the model (5.4), approximate solutions are plotted in Figure for $M=100, k=0.02$ according to the initial condition $h(x)=\cos(\pi x)$ with the two perturbation parameters taking the values $\nu=5, \mu=3$, and $a)$ $\alpha=0.05, b)$ $\alpha=0.15$, $c)$ $\alpha=0.85, d)$ $\alpha=0.95$ [8]

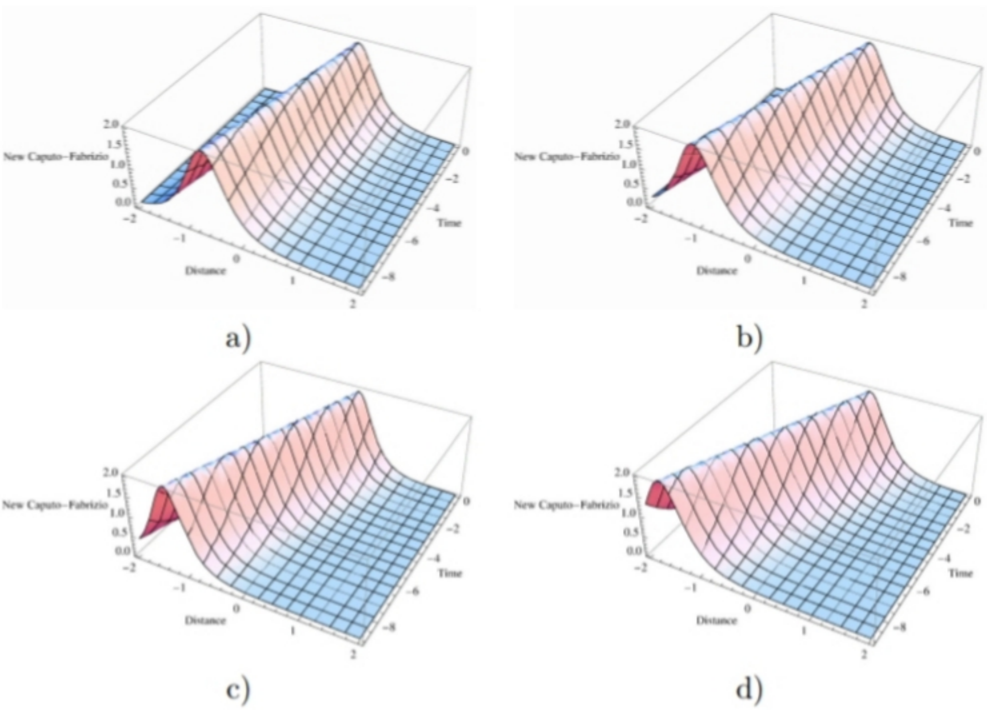

Figure 5.1: Simulation of approximation of CFFD for solution $f(x,t)$

Some notes about what we got during our study [6]

It has been shown that it is possible to extend the analysis of the Korteweg-de Vries-Burgers equation with two perturbations levels to the concepts of fractional differentiation, using the newly introduced Caputo-Fabrizio time fractional derivative with no singularity.
It has set conditions on turbulence parameters η, λ and the derivative order α, The form is below
It admits to a unique and ongoing solution. Numerical approximations have been provided, clearly showing similar behavior of the solutions for closely different values of the parameters involved.

Conclusion

• In this note, we studied the new definition of fractional derivative to Caputo-Fabrizio and his applications to differential equations of fractional order.

• As a future work, we planning to use the properties presented in this work to generalize this definition to Caputo-Fabrizio fractional differential equations and fractional differential calculus remains among the important problems that necessitate much more research.

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Abstracte

Capito-Fabrizio 's definition of fractional derivation is one of the latest definitions to improve derivation fractional derivation

In this memory, this new definition has been applied to some Cauchy fractional problems and some boundary values problems. We treat linear cases with Laplace transform and nonlinear cases by some fixed point theories.

Keywords and phrases: Capito-Fabrizio fractional derivative, Laplace transform, fixed point,

numerical approximation.

Résumé

La définition de Caputo-Fabrizio de la dérivation fractionnaire est l'une des dernières définitions pour améliorer la dérivation fractionnaire.

Dans cette mémoire, cette nouvelle définition a été appliquée à certains problème fractionnaires de Cauchy et certains problèmes aux limites.

Nous avons traité des cas linéaire par la transformée de Laplace et des cas non linéaires par des certaines théorèmes de point fixe.

Mots clés et expressions: dérivé fractionnaire Caputo-Fabrizio, transformée de Laplace, point fixe, approximation numérique .

الملخص

يعتبر تعريف كابيتوفابريزيو للإشتقاق الكسري من أحدث التعاريف لتحسين الإشتقاق الكسري، في هذه المذكرة تم تطبيق هذا التعريف على بعض مسائل كوشي الكسرية وبعض مسائل القيم الحدية، حيث عالجنا الحالات الخطية بواسطة تحويل لابلاس والحالات الغير الخطية بواسطة بعض نظريات النقطة الثابتة.

الكلمات والعبارات الدالة :المشتق الكسري لكابيتو فابريزيو, تحويل لابلاس, النقطة الثابتة، التقريب العددي .