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Using a Direct Multiple Shooting Method to an Optimal control problem

Direct Multiple Shooting Method

**Abbes Lounis · Philippe Marthon ·
Louadj Kahina · Aidene Mohamed**

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Abstract A multiple direct fire method is considered to solve optimal control problems. The multiple direct multiple shooting method is a numerical method for solving limit value problems. The method divides the interval over which a solution is sought into several smaller intervals, solves an initial value problem in each of the smaller intervals, and imposes additional matching conditions to form a solution over the entire interval. This method transforms an optimal control problem into a non-linear programming problem. To solve the latter problem, the zeros of the Lagrange Jacobian are computed using Newton's method. Then, this method is illustrated by a numerical example and finally.

Keywords Optimal control · Multiple Shooting Method · Non-Linear Programming Problem · Newton's method.

1 Introduction

Optimal control theory analyzes how to optimize dynamic systems with different criteria: reaching an objective in a minimum of time or energy, maximizing the efficiency of an industrial process for example. This involves optimizing both time-independent parameters and time-dependent control variables. Optimal control was born in the 1950s, with the work of the school of L.S.

A. Lounis
Laboratoire de Conception et Conduite des systèmes de Production, UMMTO
E-mail: lounis.bes85@yahoo.fr

P. Marthon
IRIT-ENSEEIHT. Toulouse, France.
E-mail: philippe.marthon@enseeiht.fr

· K.Louadj de Conception et Conduite des systèmes de Production, UMMTO
E-mail: louadj_kahina@yahoo.fr · Laboratoire de Conception et Conduite des systèmes de Production, UMMTO
E-mail: aidene@ummto.dz

Pontriaguine and R. Bellman [6]. Pontriaguine has obtained the conditions of optimality of a trajectory, which has made it possible to solve in an explicit or numerical way many problems, in particular of aerial or spatial trajectories. Bellman stated the the principle of dynamic programming, which also applies in the discrete or stochastic framework. The theoretical and algorithmic deepening of the theory, and the improvement of the means of calculation, make it possible to easily solve a large number of applied problems..
Mobile robot motion planning is when several robots move to the same waypoint, causing collision, blockage and deadlock. We have defined this type of dynamic stopping of a system caused by a waypoint conflict as a situation problem and the wait is due to the fact that, in order to solve this problem, a robot must wait for the others to pass through the waypoints first or the robots must wait for the planner to resume the trajectories.
The article analyzes the first and second order optimality conditions, and the ways to solve them, by time discretization, the shooting algorithm, or dynamic programming.
The aim of this paper is to solve optimal control problems using a Direct Multiple Shooting method, namely all-at- once approach, was presented by Bock in 1984 and developed recently by Diehl, Sager Scholder, and others (see [16]). Continuity of state trajectory between multiple shots is only necessary to solve the problem. This method allows to transform an optimal control problem into a non-linear control problem. programming problem. Constraints that are differential will be transformed into a system of ordinary nonlinear algebraic equations. The programming of the non-linear equations is as follows combined using the Lagrange multiplier. Finally, a Lagrangian function that represent the original optimal control problem is obtained. This problem, whose lagrangian function is defined to solve the zeros of this function, we use Newton which have been implemented in Matlab programming language.
This paper is structured as follows. In section 2, Direct Shooting method and to check the method, illustrate it with a numerical example and the results given by Matlab software. Finally, we finished by conclusion.

2 Direct Multiple Shooting method

Let consider the following optimal control problem

$$minJ = \int_0^{t_f} F(x(t), u(t), t)dt + g(x(t_f)), \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \tag{2}$$

$$x(t_f) = x_f, \tag{3}$$

$$(x(t), t) \in \mathbf{R}^2 \tag{4}$$

$$u(t) \in \mathbf{R} \tag{5}$$

The equation (2) is an ordinary differential equation with boundary condition (12). By implementing the direct method, the domain interval t is partitioned into some sub-intervals. Let

$$t_0 = 0 < t_1 < ... < t_m = t_f.$$

then, the solution of state equation (2) is determined at every sub-interval $[t_k, t_{k+1}[$ for $k = 0, 1, 2, ..., m-1$.
For $k = 0, 1, 2, ..., m-1$, let $x(t_k) = x_k$ be the given initial value for the state equation's and $u(t_k) = u_k$ at every interval $[t_k, t_{k+1}]$. By using Euler method, the value $x(t_{k+1})$ is obtained as the following formulation.

$$x(t_{k+1}) = x_k + \Delta t_{k+1} f(x_k, u_k, t_k) \tag{6}$$

where $\Delta t_{k+1} = t_{k+1} - t_k$.
The above equation is the solution of the state equation whose value will be found at every sub-interval simultaneously. However, to make the final solution function continuous at interval $[0, t_f]$, it is defined that the initial value x_{k+1} at every sub-interval is the same as the boundary value $x(t_{k+1})$ at the preceding sub-interval.

$$\begin{aligned} x(t_{k+1}) &= x_{k+1}, \\ x(t_{k+1}) - x_{k+1} &= 0, \\ x_k + \Delta t_{k+1} f(x_k, u_k, t_k) - x_{k+1} &= 0 \end{aligned} \tag{7}$$

Hence, The problem (1)-(5) is equivalent to the following problem

$$minJ = \sum_{k=0}^{m-1} F(x_k, u_k, t_k) \Delta t_{k+1} \tag{8}$$

subject to

$$\begin{aligned} &x_0 + \Delta t f(x_0, u_0, t_0) - x_1 = 0 \\ &x_1 + \Delta t f(x_1, u_1, t_1) - x_2 = 0 \\ &. \\ &. \\ &. \\ &x_{m-2} + \Delta t f(x_{m-2}, u_{m-2}, t_{m-2}) - x_{m-1} = 0 \\ &x_{m-1} + \Delta t f(x_{m-1}, u_{m-1}, t_{m-1}) - x_f = 0. \end{aligned} \tag{9}$$

This preceding problem is now a constrained non-linear programming problem which can be solve by applying Lagrange multiplier method.

Let $p = (p_1, p_2, ..., p_m)$ be a vector of the Lagrange multiplier, then the previous constrained non-linear programming system can be reformulated as follow.

$$L = \sum_{k=0}^{m-1} F(x_k, u_k, t_k) \tag{10}$$

$$+ p^T \begin{bmatrix} x_0 + \Delta t f(x_0, u_0, t_0) - x_1 \\ x_1 + \Delta t f(x_1, u_1, t_1) - x_2 \\ . \\ . \\ . \\ x_{m-1} + \Delta t f(x_{m-1}, u_{m-1}, t_{m-1}) - x_f \end{bmatrix} \tag{11}$$

and the first order condition are

$$\nabla_x L(x, u, p) = 0 \tag{12}$$

$$\nabla_u L(x, u, p) = 0 \tag{13}$$

$$\nabla_p L(x, u, p) = 0 \tag{14}$$

Where $x = (x_0, x_1, ..., x_m)$ and $u = (u_0, u_1, ..., u_{m-1})$.
The first order conditions (12)-(14) give a system of non-linear algebraic equations. The final solution of the system of this problem which originally the optimal control can be obtained by solving the system of equations (12)-(14). Finally, the Newton's method is applied to solve this system of non-linear algebraic equations to see how the Newton method which is implemented in a code program works to this problem.
To illustrate this method, given an example to show efficiency of the method.

Let us consider a following problem [1]

$$minJ = \int_0^1 (x^2 + u^2) dt \tag{15}$$

subject to,

$$\dot{x}(t) = (x - tu)^2 \tag{16}$$

$$x(1) = 2, (x(t), t) \in R^2 \tag{17}$$

$$u(t) \in R \tag{18}$$

Here, the domain interval t is divided into four partitions by step size $\Delta t = t_f/m$. Here, the optimal control problem (15)-(18) can be transformed into the system below.

$$minJ = \sum_{k=0}^{3} (x_k^2 + u_k^2) \Delta t \tag{19}$$

$$x_0 + \Delta t x_0^2 - x_1 = 0 \tag{20}$$

$$x_1 + \Delta t (x_1 - 0.25 u_1)^2 - x_2 = 0 \tag{21}$$

$$x_2 + \Delta t(x_2 - 0.5u_2)^2 - x_3 = 0 \quad (22)$$

$$x_3 + \Delta t(x_3 - 0.75u_3)^2 - x_4 = 0 \quad (23)$$

. . .

$$x_{m-1} + \Delta t(x_{m-1} - m\Delta t u_3)^2 - x_f = 0 \quad (24)$$

The problem is implemented using Matlab language. Let consider the initial condition
$x = 10 * ones(m, 1)$, $u = 10 * ones(m, 1)$, $p = 10 * ones(m, 1)$
and the solution is given by

$$xfsol = x(m) + \Delta t * (x(m) - (m-1)\Delta t * u(m))^2$$

After implementation of this method in Matlab, we consider m is variable. The problem of convergence of this iterative process is solved by a result of the book of Ortega and Rheinboldt [15]; in fact if the discretization h_{ij} are small and tend to zero, it is ensured that the convergence process.

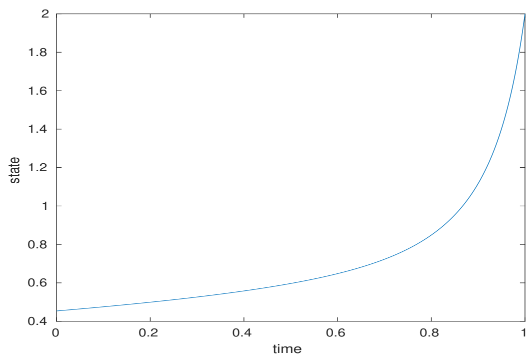

Fig. 1 The state x

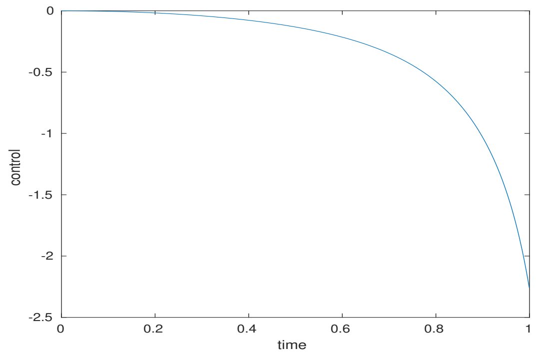

Fig. 2 The control u

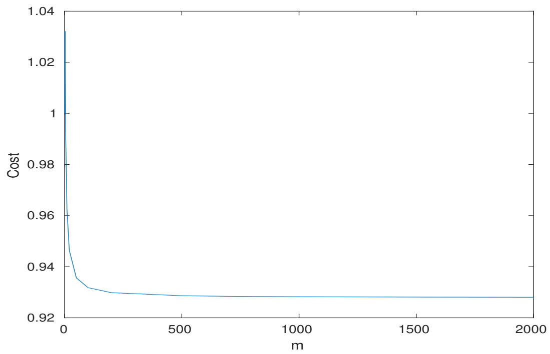

Fig. 3 The cost J

The figure (3) shows the convergence of the method, and optimum is reached.

3 Conclusion

By using multiple Shooting method, optimal control problems can be transformed into a non-linear programming problem which can be solved by using Lagrange multiplier method. The Lagrangian function solved by Newton Method's. This last method, The problem of convergence the step the fast convergence, and a very short time of computation.

References

1. Said Munzir, Vera Halfiani and Marwan. *An optimal control solution using multiple shooting method.* Bulletin of Mathematics.Vol4 N2(2012),pp.143-180.

2. H.G.Bock and K.J.Plitt.*A multiple Shooting Algorithm for direct solution of optimal control problems*. IFAC Proceeding, Volume 17, Issue 2, July 1984,pp.1603-1608.
3. Efstathios Bakolas-Panagiotis Tsiotras, Optimal Synthesis of the Zermelo-Markov-Dubins, Problem in a constant Drift Field, J.Optimal theory applications, Springer, 156:469-492. 2013.
4. David G. Hull, Optimal control theory for applications, Springer-Verlag, 2003.
5. J.M.Ortega and W.C.Rheinboldt, Iterative solution of nonlinear equations in several variables. Academic Press, New York, 1970.
6. E.Trelat, Contrôle optimal: théorie et applications, Vuibert, collection Mathématiques Concrètes, 2005.
7. Suresh P.Sethi, Gerald L.Thompsonn Optimal control theory, applications to management science and economics, second edition, 2000.
8. R. E. Bellman. *Dynamic programming*. Princeton University Press, Princeton, NJ, 1963.
9. F. Demim, A.Nemra, K.Louadj. *Solution of an optimal control problem of the robot with Bellman Principle's*. Conférence Internationale en Sciences et Technologies Electriques, October 26-28 , CISTEM'16, Morocco, Marakech, 2016.
10. F.Demim, K.Louadj, M.Aidene, A. Nemra. *Solution of an Optimal Control Problem with Vector Control using Relaxation Method*, Automatic Control and System Engineering Journal, Volume 16, Issue 2, ISSN 1687-4811, ICGST LLC, Delaware, USA, 2016.
11. K.Louadj, P.Spitéri, M.Aidene, F.Messine. *An optimal control problem with free final time for aircraft flight with wind*. Colloque sur l'optimisation et les systèmes d'information (COSI 2014), Bejaia, 08/06/2014-10/06/2014.
12. K.Louadj, P.Spitéri, M.Aidene, F.Messine. *Optimal command for the control of the air navigation of an aircraft*. The Second International Conference on Electrical Engineering and Control Applications, Constantine, 18/11/2014-20/11/2014.
13. S.Titouche, P.Spiteri, F. Messine, M.Aidene. *Optimal control of a large thermic*, Journal Control System, Vol.25, pp.50-58, January 2015.
14. F.Demim, A. Nemra, K.Louadj, M.Hamerlain. *Simultaneous localization, mapping, and path planning for unmanned vehicle using optimal control*, Advances in Mechanical Engineering, 2018, Vol. 10(1) 1-25.
15. J.M.Ortega, W.C.Rheinboldt. *Iterative solution of nonlinear equations in several variables*, Academic Press, New York, 1970.
16. H.G.Bock, K.J.Plitt. A multiple shooting algorithm for direct solution optimal control problems. in proceeding of the 9th IFAC world Congress. Budapest, Pergamon Press. 1984.pp.243-247.
17. L.Pontryagin et al, *Mathematical theory of optimal processes*, Eds Mir, Moscou 1974.