Algerian People's Democratic Republic
Ministry of Higher Education and Research Scientist

University of Chahid Hama Lakhdar El Oued

Faculty of Exact Sciences
Department of Mathematics

Present for the Master's Degree

Option: Fundamental mathematics

THE THESIS:

Fractional Differential Equations With Initial Conditions at Inner Points

---

Presented by:

BEGGAT Roufeida

BEGGAT Aicha

The jury composed of:

.	DR. Beloul Said	Univ El-Oued	President
.	Professor Nisse Lamine	Univ El-Oued	Promoter
.	Dr. Nisse Khadidja	Univ El-Oued	Examiner

Academic year: 2020/2021

GENERAL NOTATION

- $\mathbb{R}$ The set of real numbers.
- $\mathbb{R}^+$ The set of positive real numbers.
- $\mathbb{R}^n$ An n-dimensional real vector space constructed over the field of reals.
- $\mathbb{N}$ Set of natural numbers.
- $\mathbb{C}$ The set of complex numbers.
- $[a, b)$ Semi-open interval in $\mathbb{R}$ extremities a and b.
- $\mathcal{C} = \mathcal{C}(K, F)$ Set of continuous functions from K to F.
- $\mathcal{C}^n = \mathcal{C}^n([a, b])$ Function space n-time continuously differentiable in $[a, b]$.
- $D(U)$ Operator's domain U.
- $|\,.\,|$ The absolute value of a real number or modulus of a complex number.
- $\Gamma(.)$ Euler's gamma function.
- $B(.,.)$ Beta function.
- $E_{\alpha,\beta}\ (,)$ Mittag-Leffler Function.
- I^α Integration of ordre α.
- ${}^{C}D^\alpha$ Caputo derivative of order α.
- ${}^{RL}D^\alpha$ Riemann-Liouville derivative of order α.
- LT Laplas transformation.
- $\alpha(.)$ compactness measure.

Contents

1 Introduction to fractional calculus 6

1.1 Special functions . . . 6

1.1.1 Euler's Function . . . 6

1.1.2 Mittag-Leffer function . . . 9

1.2 Fractional Integrals and Derivatives . . . 9

1.2.1 Riemann-Liouville Integrals . . . 9

1.2.2 Fractional Derivatives . . . 11

1.2.3 Caputo's Approach . . . 14

1.3 Introduction to The Fractional Differential Equations: . . . 15

1.3.1 Fixed Point Theorems: . . . 15

1.3.2 The Existence and Uniqueness Theorem for Initial Value Problems 17

1.3.3 Existence and Uniqueness for the Caputo Problem [6] . . . 19

2 The Ordinary Fractional Differential Equation 22

2.1 Cauchy Problems via Measure of Noncompactness Method . . . 22

2.1.1 Existence . . . 23

2.2 Cauchy Problems via Topological Degree Method . . . 26

2.2.1 Qualitative Analysis . . . 26

3 Fractional Differential Equations with Initial Conditions at Inner Points 30

3.1 Existence and Uniqueness Results . . . 30

Introduction

The theory of differential and integral operators of non-integer order (fractional order), and in particular that of differential equations associated with such operators, is of different nature than in the ordinary case. Even though the first steps of the theory itself date back to the first half of the nineteenth century, the subject only really came to life over the last few decades.

"*In the letters to J. Wallis and J. Bernulli (in 1697) Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of* n *the definition could be the following:*

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx}$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n*, for example* $\frac{d^{m/n}x}{dx^{m/n}}$ *Taking* $m = 1$ *and* $n = \frac{1}{2}$*, Euler obtained*:

$$\frac{d^{1/2}x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} \quad \left(= \frac{2}{\sqrt{\pi}}x^{1/2}\right)$$

" [11]

In the past sixty years, the fractional calculus as a main tool in the study of fractional order differential equations, has played a very important role in various fields such as physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, aerodynamics, experimental data processing, etc. There are many works on the existence, uniqueness and qualitative behavior of solutions of fractional differential equations. For more details on this subject, we refer to the monograph of Kilbas et al [5] . So far the theory of initial value problems associated with fractional differential equations is quite developed when the initial values are taken at the end-points of the definition interval of the derivative. In the theory of ordinary differential equations the position of the point of the initial value in the interval of definition does not play a determining role. But in the fractional case the situation is fundamentally different. It is this aspect of the theory that has motivated the study presented in this thesis.

Here is an example that illustrates our topic (see [1]). Consider the following two initial value problems of fractional order,

$$(P1) \qquad \begin{cases} {}^cD_0^\alpha y_1(x) = x^2, \quad x \geq 1 \in [0\,,2[\\ y_1\,(1) = \frac{2}{\Gamma(3+\alpha)} \end{cases}$$

and

$$(P2) \qquad \begin{cases} {}^cD_1^\alpha y_2(x) = x^2, \quad x \geq 1 \in [0\,,2[\\ y_2\,(1) = \frac{2}{\Gamma(3+\alpha)} \end{cases}$$

where $\alpha \in]0\,,1[$. It is well known that the solutions are given by

$$\begin{cases} y_1(t) = y_1(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-x)^{\alpha-1} x^2 dx, \quad t \geq 1 \in [0\,,2[\\ where,\ \ y_1(0) = \frac{2}{\Gamma(3+\alpha)} - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^1 (1-x)^{\alpha-1} x^2 dx \end{cases}$$

and

$$y_2(t) = \frac{2}{\Gamma(3+\alpha)} + \frac{1}{\Gamma(\alpha)} \int_1^t (t-x)^{\alpha-1} x^2 dx, \quad t \geq 1 \in [1\,,2[.$$

Thus, by a direct computation using integration by parts, we should end up with

$$y_1(t) = \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)},$$

and

$$y_2(t) = \frac{2}{\Gamma(3+\alpha)} + \frac{(t-1)^\alpha}{\Gamma(1+\alpha)} + \frac{2(t-1)^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2(t-1)^{2+\alpha}}{\Gamma(3+\alpha)}.$$

A verification by a simple numerical computation (take for example $\alpha = 0,5$) shows that the solutions y_1 and y_2 are different on the interval $[1\,,2]$. This shows, contrary to ordinary differential problems, that the position of the initial value point is fundamentally determinant of the solution when the order of the differential equation is not an integer.

Presentation of the thesis

This thesis is broken down into three chapters:

Chapter 1 an introduction to the theory of special function, the Gamma, Beta and mittag-Luffer function, which play the most important role in the theory of fractional derivatives and fractional differential equations, we recall the different fixed point theorems, and some results of existence and uniqueness for initial value problems, and existence and uniqueness for the Caputo Problem.

Chapter 2 in this chapter we discuss the Cauchy problem of fractional ordinary differential equations in Banach spaces under hypotheses based on Carathéodory condition. The tools used include some classical and modern nonlinear analysis methods such as fixed point theory, measure of noncompactness method, topological degree method and Picard operators technique, etc.

Chapter 3 We present the results of existence and uniqueness of fractional differential equation with inner initial value, with the Caputo derivative.

Chapter 1

Introduction to fractional calculus

1.1 Special functions

1.1.1 Euler's Function

Gamma Function

We start by considering the Gamma function, or second order Euler integral, denoted $\Gamma(.)$, Function Gamma (Γ) is defined as (see [3] [4] [6]) :

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx$$

This integral converges in the right half of the complex plane Re(p) ¿ 0.

Some properties of Gamma function:

The gamma function is that it satisfies the following properties:

1. $\Gamma(p+1) = p\Gamma(p)$
2. The following relations are also valid:

$$\Gamma(p+n) = (p+n-1)...(p+1)p\Gamma(p)$$

In particular : $\Gamma(1) = 1$, we deduce that: $\Gamma(n+1) = n!$

3. $\Gamma(p) = \dfrac{\Gamma(p+1)}{p}$

4. For $p > 1$:

$$\frac{p+1}{\Gamma(p+1)} = \frac{p+1}{p\Gamma(p)} < \frac{2}{\Gamma(p)}$$

5. The following particular values for Γ function :

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(0_+) = +\infty$

$\Gamma(n+\frac{1}{2}) = \frac{(2n!)}{2^{2n}n!}\sqrt{\pi}, \forall n \in \mathbb{N}$

Proof (of the 1st property)

$$\Gamma(p+1) = \int_0^\infty e^{-x}x^p dx = -\left[e^{-x}x^p\right]_0^\infty + p\int_0^\infty e^{-x}x^{p-1}dx = p\Gamma(p)$$

The graph of Gamma function :

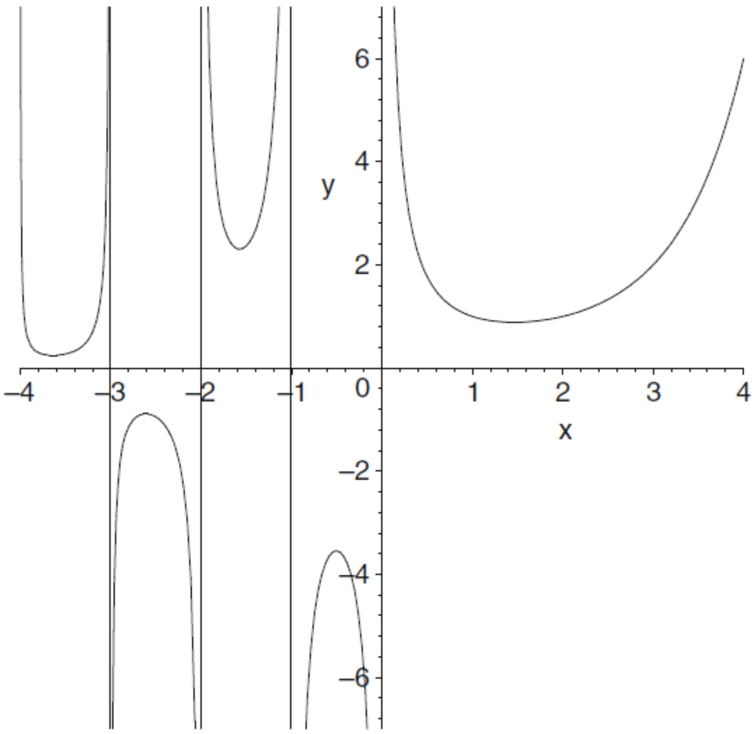

Figure 1.1: Gamma Function

Beta Function

The Beta function, or the first order Euler function, can be defined as (see [3] [4] [6]):

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

where $Re(p) > 0$ and $Re(q) > 0$.

Proposition (see [**?**])

The relation between Beta Function and Gamma Function gives by :

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1.1}$$

$Re(p) > 0$, $R(eq) > 0$.

Properties [4]

In the following we will enumerate the basic properties of the Beta function:

- For every $p > 0$ and $q > 0$, we have:

$$B(p,q) = B(q,p)$$

- For every $p > 0$ and $q > 1$, the Beta function B satisfies the property:

$$B(p,q) = \frac{q-1}{p+q-1}B(p,q-1)$$

- For every $p > 0$, and for $n \in \mathbb{N}$:

$$B(p,n) = B(n,p) = \frac{1 \cdot 2 \cdot 3 \ldots (n-1)}{p(p+1)\ldots(p+n)}$$

  and also:

$$B(p,1) = \frac{1}{p} \ .$$

1.1.2 Mittag-Leffer function

In this section we introduce the one- and two-parameter Mittag-Leffler functions, denoted as $E_{\alpha()}$ and $E_{\alpha,\beta()}$, respectively. The one-parameter Mittag-Leffler function (E_α), is defined as (see [3] [4] [6]) :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad Re(\alpha) > 0.$$

For particular values of $\alpha = 1$:

$$E_1(z) = \exp(z)$$

The two parameter Mittag-Leffer function $E_{(\alpha,\beta)}$, is defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad Re(\alpha) > 0, Re(\beta) > 0, \beta \in \mathbb{C}$$

The graph of Mittag-Leffer function:

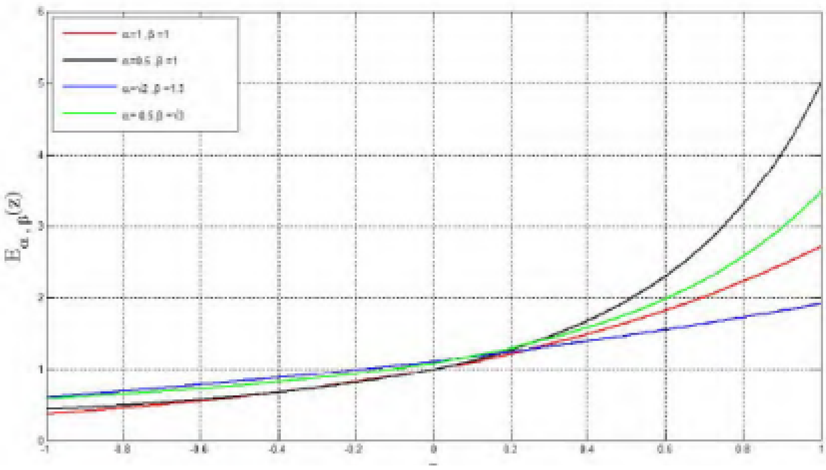

Figure 1.2: Mittag-Leffer function

1.2 Fractional Integrals and Derivatives

1.2.1 Riemann-Liouville Integrals

Definition

Let $n \in \mathbb{R}_+$. the operator J_a^n, defined on $L^1\,[0,1]$

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)dt \tag{1.2}$$

for $a \leq x \leq b$, is called fractional the Riemann-Liouville fractional integral of order n. For $n = 0$, we set $J_a^0 := I$, the identity operator.

Theorem [6]

Let $f \in L^1[a, b]$ and $n > 0$. The the integral $J_a^n f(x)$ exists for almost every $x \in [a, b]$. Moreover, the function $J_a^n f(x)$ itself is also an element of $L^1[a, b]$.

Theorem

Let $m, n \geq 0$ and $\phi \in L^1[a, b]$, Then

$$J_a^m J_a^n \phi = J_a^{m+n} \phi \tag{1.3}$$

holds almost everywhere on $[a, b]$, If additionally $\phi \in C[a, b]$ or $m + n \geq 1$, Then the identity holds everwhere on $[a, b]$.

Example

The Riemann-Liouville integral of order n of the function : $f(x) = (x - a)^p$

$$J_a^n (x - a)^m = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{n-1} (x - a)^p dt$$

we put : $t = a + s(x - a)$

$$I_a^n (x - a)^m = \frac{1}{\Gamma(\alpha)} (x - a)^{p+n} \int_0^1 (1 - s)^{n-1} s^p ds = \frac{1}{\Gamma(\alpha)} (x - a)^{p+n} B(p + 1, n)$$

The proposition (1.1) gives:

$$I_a^n (x - a)^p = \frac{\Gamma(p + 1)}{\Gamma(p + n + 1)} (x - a)^{p+n}$$

1.2.2 Fractional Derivatives

Riemann-Liouville Derivative:

Let f be an integrable function $f\,[a,b] \to \mathbb{R}$.

Definition [3]

Let $n \in \mathbb{R}_+$ and $m = [n]$. The operator D_a^n, defined by

$$D_a^n f := D^m J_a^{m-n} f \tag{1.4}$$

is called the Riemann-Liouville fractional differential operator of order n.

For n=0, we set $D_a^0 = I$, the identity operator.

Example

1. Let $f(x) = (x-\alpha)^p$, with $p > -1$, For $\alpha \geq 0$ and $n-1 \leq \alpha \leq n$

$$D_a^\alpha f(x) = D^{n-\alpha} f(x) = \frac{\Gamma(p+1)}{\Gamma(p+n-\alpha+1)} D^n (x-a)^{n-\alpha+p}$$

   Then, for $(\alpha - p) \in \{1, 2, ...n\}$: $D_a^\alpha (x-a)^{\alpha-j}, \qquad j \in \{1, 2, .., n\}$

2. In particular, if $p = 0$ and $\alpha > 0$, the Reimann-Liouville fractional derivative of constant function $f(x) = C$ is not zero, its value:

$$D_a^\alpha C = \frac{C(x-\alpha)^{-\alpha}}{\Gamma(1-\alpha)}$$

Proposition

Let $\alpha, \beta > 0$, such that $\alpha \in \,]n-1, n[\,$, $\beta \in \,]m-1, m[$, Then :

1 For all $t \in [a,b]$, $f \in L^1\,[a,b]$.

$$D_a^\alpha (I_a^\alpha f(t)) = f(t) \tag{1.5}$$

2 If $0 < \alpha \leq \beta$, then for all $f \in L^1[a,b]$:

$$(D_a^{\beta}(D_a^{\alpha} f))(x) = (I^{\alpha-\beta} f)(x) \tag{1.6}$$

3 If $0 < \alpha \leq \beta$, and if the fractional derivative exists, then:

$$D_a^{\beta}(I^{\alpha} f)(x) = (D_a^{\beta-\alpha} f)(x) \tag{1.7}$$

4

$$\left[I_a^{\alpha}(D_a^{\alpha} f)\right](x) = f(x) - \Sigma_{j=0}^{m-1} \frac{(x-1)^{j-m+\alpha}}{\Gamma(j-m+\alpha+1)} \{ \lim_{x \longrightarrow a^+} \left[(\frac{d}{x})^j I_a^{m-\alpha} f \right](x) \} \tag{1.8}$$

Caputo Derivative

Definition: [3]

Let $\alpha > 0$, $n = [\alpha]$. The Caputo derivative operator of order α is defined as:

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(u)(t-u)^{n-\alpha-1} du$$

For $\alpha = 0$ we introduce the notation ${}_aD_t^{\alpha} f(t) = D^{\alpha} f(t)$

Example: [7]

$f(x) = (x-\alpha)^{\beta}$ such that $\beta > 0$

1. if $\beta \in \{1, 2, 3...n-1\}$ then:

$$^{C}D_{a}^{\alpha} f(x) = 0$$

2. If $\beta > n-1$ then:

$$^{C}D_{a}^{\alpha}(x-\alpha)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-\alpha)^{\beta-\alpha}$$

In particular, if f is constat over $[a,b]$ then:

$$^{C}D_{a}^{\alpha} f(x) = 0$$

Proposition:

Caputo fractional derivatives verify these properties:

1. $^{C}D_{a}^{\alpha}\left[I_{a}^{\alpha} f\right] = f$

2. If $^{C}D_{a}^{\alpha} f = 0$ then $f(x) = \Sigma_{j=0}^{n-1} c_j (x-a)^j$, such that $c_j \in \mathbb{R}$.

3. $I_{a}^{\alpha}\left[^{C}D_{a}^{\alpha} f\right](x) = f(x) - \Sigma_{j=0}^{n-1} \dfrac{(x-a)^j}{j!} f^{(j)}(a)$

Lemma [3]

Let f be a continue function over $[a,b]$ et $\alpha > 0$.

$$\lim_{x \longrightarrow a^{+}} (I_{a}^{\alpha} f)(x) = 0$$

Proof

$$\begin{aligned} \mid (I_{a}^{\alpha} f)(x) \mid &\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-\alpha)^{\alpha-1} \mid f(t) \mid dt \\ &\leq \frac{\parallel f \parallel_{\infty}}{\Gamma(\alpha)} \int_{a}^{x} (x-a)^{\alpha-1} dt \\ &\leq \frac{\parallel f \parallel_{\infty}}{\Gamma(\alpha+1)} (x-a)^{\alpha} \end{aligned}$$

Corollary [4]

If $0 < \alpha < 1$ and f of class C^1 then

$$(I_a^\alpha \circ^R D_a^\alpha)f = f \text{ and } (^C D_a^\alpha \circ I_a^\alpha)f = f$$

Corollary

If $\alpha \leq 0$, $\beta \leq 1$ with $\alpha + \beta \leq 1$, f of class C^1 then

$$(^C D_a^\alpha \circ^C D_a^\beta)f = (^C D_a^{\alpha+\beta})f = (^C D_a^\beta \circ^C D_a^\alpha)f$$

Proof

It is easy to see that

$$\begin{aligned}
(^C D_a^\alpha \circ^C D_a^\alpha)f &= (I_a^{1-\alpha} \circ \tfrac{d}{dx} \circ I_a^{1-\beta} \circ \frac{d}{dx})f \\
&= (I_a^{1-\alpha-\beta} \circ \underbrace{I_a^\beta \circ \frac{d}{dx}} \circ I_a^{1-\beta} \circ \frac{d}{dx})f \\
&(I_a^{1-\alpha-\beta} \circ \underbrace{^C D_a^{1-\beta} \circ I_a^{1-\beta}} \circ \frac{d}{dx})f \\
&= (I_a^{1-\alpha-\beta} \circ \frac{d}{dx})f \\
&=^C D_a^{\alpha+\beta} f.
\end{aligned}$$

1.2.3 Caputo's Approach

It turns out that the Riemann–Liouville derivatives have certain disadvantages when trying to model real-world phenomena with fractional differential equations. We shall therefore now discuss a modified concept of a fractional derivative. As we will see below when comparing the two ideas, this second one seems to be better suited to such tasks. [4]

lemma [7]

Let $\alpha \geq 0$, $n = [\alpha] + 1$, if f has $(n-1)$ derivatives on a, and if $^R D_a^\alpha$ exists, then:

$$(^C D_a^\alpha f)(x) =^R D_a^\alpha \left[f(x) - \Sigma_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right]$$

almost everywhere in $[a, b]$.

Proof

According to the definition we have:

$$\begin{aligned} {}^{R}D_a^{\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] &={}^{R} D^n I_a^{n-\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] \\ &=\frac{d^n}{dx^n}\int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left[f(t)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]dt \end{aligned}$$

Using integration by part we get :

$$\begin{aligned} I_a^{n-\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] &=\int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}\left[f(t)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k dt\right]dt \\ &=I_a^{n-\alpha+1}D\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k}(x-a)^k\right] \end{aligned}$$

In the same way for $n-1$ times:

$$\begin{aligned} I_a^{n-\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] &=I_a^{n-\alpha+n}D^n\left[f(x)-\Sigma_{k=0}^{n-n}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] \\ &=I_a^n I_a^{n-\alpha+1}D^n\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{(x-a)^k}\right]. \end{aligned}$$

Such that $\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k$ is a polynomial of degree $n-1$, then:

$$\begin{aligned} I_a^{n-\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{k}(a)}{k!}(x-a)^k\right] &=I_a^n I_a^{n-\alpha+1}D^n f(x). \\ {}^{R}D_a^{\alpha}\left[f(x)-\Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right] &=D^n I_a^n I_a^{n-\alpha+1}D^n f(x) \\ &=I^{n-\alpha+1}D^n f(x) \\ &=({}^{C}D_a^{\alpha}f)(x) \end{aligned}$$

Almost wherever on $[a,b]$.

1.3 Introduction to The Fractional Differential Equations:

1.3.1 Fixed Point Theorems:

The proofs of various existence and uniqueness theorems throughout this text have been based on classical theorems asserting existence or uniqueness of fixed points of certain operators.

The first of these theorems is the following generalization of Banach's fixed point theorem.

Theorem (Weissinger's Fixed Point Theorem):[6]

Assume (U, d) to be a nonempty complete metric space, and let $\alpha_j \geq 0$ for every $j \in \mathbb{N}$ and such that $\Sigma_{j=0}^{\infty}\alpha_j$ converges. Furthermore, let the mapping $A : U \longrightarrow U$ satisfy the inequality :

$$d(A^j u, A^j v) \leq \alpha_j d(u, v)$$

for every $j \in \mathbb{N}$ and every $u, v \in U$. Then, A has a uniquely determined fixed point u^*. Moreover, for any $u_0 \in U$, the sequence $(A_j u_0)_{j=1}^{\infty}$ converges to this fixed point u^*.
An immediate consequence is

Corollary (Banach's Fixed Point Theorem):[6]

Assume (U, d) to be a nonempty complete metric space, let $0 \leq \alpha < 1$, and let the mapping $A : U \longrightarrow U$ satisfy the inequality

$$d(Au, Av) \leq \alpha d(u, v)$$

for every $u, v \in U$. Then, A has a uniquely determined fixed point u^*. Furthermore, for any $u_0 \in U$, the sequence $(A_j u_0)_{j=1}^{\infty}$ converges to this fixed point u^*.

Moreover we also used a slightly different result that asserts only the existence but not the uniqueness of a fixed point. Here we may work with weaker assumptions on the operator in question.

Theorem (Schauder's Fixed Point Theorem):[6]

Let (E, d) be a complete metric space, let U be a closed convex subset of E, and let $A : U \longrightarrow U$ be a mapping such that the set $Au : u \in U$ is relatively compact in E. Then A has at least one fixed point.

Definition

Let (E, d) be a metric space and $F \subseteq E$. The set F is called relatively compact in E if the closure of F is a compact subset of E.

Theorem (Arzel'a–Ascoli)[7]

Let $F \subseteq C\,[a,b]$ for some $a < b$, and assume the sets to be equipped with the Chebyshev norm. Then, F is relatively compact in $C\,[a,b]$ if F is equicontinuous (i.e. for every $\epsilon > 0$ there exists some $\delta > 0$ such that for all $f \in F$ and all $x, x^* \in [a,b]$ with $\mid x - x^* \mid < \delta$ we have $\mid f(x)f(x^*) \mid < \epsilon$) and uniformly bounded (i.e. there exists a constant $C > 0$ such that $\parallel f \parallel_\infty \leq 0$ for every $f \in F$).

Definition (Chebyshev Norm) [6]

The Chebyshev norm on a set S is:

$$\parallel f \parallel_\infty = \{|f(x)|, x \in S\}$$

where Supremum (sup) denotes the supremum.

1.3.2 The Existence and Uniqueness Theorem for Initial Value Problems

Definition [6]

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $n = [\alpha] + 1$, and $f : A \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ then

$$^R D^\alpha u(x) = f(x, u(x))$$

with the conditions:

$$^R D^{\alpha-k} u(0) = b_k \qquad (k = 1, 2, ..., n-1) \qquad lim_{z \longrightarrow 0^+} I^{n-\alpha} u(z) = b_n$$

called also Riemann–Liouville FDE.

Definition [6]

Let the FDE

$$^C D^\alpha u(x) = f(x, u(x))$$

with the initial conditional

$$u^k(0) = b_k \qquad (k = 0, 1, .., n-1)$$

Called also Caputo FDE.

Lemma [6]

Let $u(t)$ be a function with continuous derivative in the interval $I_h(0) = [0, h\}$ with values in $[u_0\eta, u_0 + \eta]$, then y(t) satisfies the Caputo type

$$D^\alpha u(t) = f(t, u(t)) \quad 0 < \alpha \leq 1 \quad t > 0$$
$$u(0) = u_0$$

if and only if it satisfies the Voltera integral,

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x, u(x))dx$$

Proof (see [6])

Let $L[u(t)] = U$ be the LT of $u(t)$. We have

$$s^\alpha U - s^{\alpha-1}u_0 = L\left[f(t, u(t))\right]$$
$$U = \frac{u_0}{s} + \frac{1}{s^\alpha} L\left[f(t, u(t))\right]$$
$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x, u(x))dx$$

Lemma (The Weierstrass Test) [6]

Suppose that $\{f_n(t)\}$ is a sequence of real functions defined on a set A, and there is a sequence of positive numbers $\{R_n\}$ satisfying:

$$\forall n > 1 \quad \forall f \in A, \quad |f_n(t)| \leq R_n, \quad \Sigma_{n=1}^{\infty} R_n < \infty$$

Then the series $\Sigma_{n=1}^{\infty} f_n(t)$ is convergent.

1.3.3 Existence and Uniqueness for the Caputo Problem [6]

Theorem

Let a Caputo FDE be

$$D^{\alpha}u(t) = f(t, u(t)) \quad 0 < \alpha \leq 1, \quad t > 0$$

with the initial condition: $u(0) = u_0$

We consider the domain: $D = [0, \eta] \times [u_0\eta, u_0 + \eta]$

on which f satisfies:

- $f(t, u)$ is continuous.
- $|f(t, u)| < M$, where $M = max_{(t,u)\in D}|f(t, u)|$, and Maximum (max) denotes the maximum function.
- $f(t, u)$ satisfy in D the Lipschitz condition in u if there is a constant K such that:

$$|f(t, u_2)f(t, u_1)| \leq K|u_2u_1|$$

Then it exists $\delta > 0$ and a function $u(t) \in C[0, \eta]$ unique for

$$\delta = min\{\eta, \left(\frac{\eta\Gamma(\alpha + 1)}{M}\right)^{\frac{1}{\alpha}}\}$$

where Minimum (min) denotes the minimum function.

Remark

In order to prove the existence of the solution we can introduce the set:

$$U = \{u \in C\,[0, \eta] : \|uu_0\| \leq \eta\}$$

and an operator A:

$$Au(t) = u0 + \frac{1}{\Gamma(\alpha)} \int_0^t (ts)^{\alpha-1} f(s, u(s))ds$$

where A has a fixed point, and U is a closed and convex subset of all continuous functions on $[0, \eta]$ equipped with Chebyshev norm.

Generally:

$$u(t) = \Sigma_{j=0}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)}(t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, u(s))}{(ts)^{1-\alpha}} ds$$

where $t > 0, n1 \leq \alpha < n$.

The technique used for proving the existence solution of the Voltera equation is often the successive approximation:

$$u_0(t) = \Sigma_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)} t^{\alpha-k}$$
$$u_i(t) = u_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u_{i-1}(\tau)) d\tau \qquad i = 1, 2..$$
$$u(t) = lim_{i \longrightarrow \infty} U_i(t)$$

Proof (see [6])

Stability

Balance point

We consider the autonomous system described by:

$$\frac{dx}{dt} = f(x), x \in \mathbb{R}^n$$

f of class C^1.

The notion of the balance point is the essential notion in the study of stability.

Definition

x_ϵ is a balance point if :

$$lim_{t \longrightarrow \infty} x(t) = x_\epsilon$$

x_ϵ verify $f(x_{\epsilon)=0}$

The stability with Caputo Derivative

Matignon studied the following differential system, involving the fractional derivative of Caputo

$$^{C}D_0^{\alpha}x(t) = Ax(t)$$

With the initial condition :

$x(0) = x_0 = (x_{1_0}, x_{2_0}, ..., x_{n_0})^T$ such that $x = (x_1, x_2...x_n)$, $\alpha \in (0,1)$ and $A \in \mathbb{R}^p \times \mathbb{R}^n$.

Definition

The autonomous system would be:

(a) Stable if for all x_0, exists $\epsilon > 0$ such that $\| x(t) \| \leq \epsilon$ for all $t > 0$.

(b) asymptotically stable if :

$$lim_{t \longrightarrow \infty} \| x(t) \| = 0$$

Theorem

The autonomous system is:

- asymptotically stable if $|arg(\lambda(A))| > \dfrac{\alpha\pi}{2}$
- stable if asymptotically stable, or those eigenvalues of the matrix which satisfy $|arg(\lambda(A))| > \dfrac{\alpha\pi}{2}$ have geometric multiplicity.

Such that $|arg(\lambda(A))|$ denotes arguments of the eigenvalues of the square matrix A.

Chapter 2

The Ordinary Fractional Differential Equation

2.1 Cauchy Problems via Measure of Noncompactness Method

In this Section, we assume that X is a Banach space with the norm $|\ .\ |$. Let $J \subset R$. Denote $C(J, X)$ be the Banach space of continuous functions from J into X. Let $r > 0$ and $C = C([r, 0], X)$ be the space of continuous functions from $[r, 0]$ into X. For any element $z \in C$, define the norm $\| z \|_{*} = sup_{\theta \in [-r,0]} | z(\theta) |$.

Consider the initial value problem (IVP for short) for fractional functional differential equation given by

$$\begin{cases} {}_0^C D_t^q x(t) = f(t, x_t), & t \in (0, a) \\ x_0 = \varphi \in \mathcal{C} \end{cases} \tag{2.1}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $0 < q < 1$, $f : [0, a) \times \mathcal{C} \to X$ is a given function satisfying some assumptions and define x_t by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$ Now, we shall discuss the existence of the solutions for fractional IVP (2.1) under assumptions that f satisfies Carathéodory condition and the condition on measure of noncompactness. .

Definition [6]

A function $x \in C([-r,T],X)$ is a solution for fractional IVP (2.1) on $[-r,T]$ for $T \in (0,a)$ if

1. the function $x(t)$ is absolutely continuous on $[0,T]$;

2. $x_0 = \varphi$;

3. x satisfies the equation in (2.1).

2.1.1 Existence

[6]

Now we start proving the existence of the solutions for fractional IVP (2.1) under the following hypotheses:

(H1) for almost all $t \in [0,a)$, the function $f(t,\cdot) : \mathcal{C} \to X$ is continuous and for each $z \in \mathcal{C}$, the function $f(\cdot,z) : [0,a) \to X$ is strongly measurable;

(H2) for each $\tau > 0$, there exist a constant $q_1 \in [0,q)$ and $m_1 \in L^{\frac{1}{q_1}}([0,a),\mathbb{R}^+)$ such that $|f(t,z)| \le m_1(t)$ for all $z \in \mathcal{C}$ with $\|z\|_* \le \tau$ and almost all $t \in [0,a)$;

(H3) there exist a constant $q_2 \in (0,q)$ and $m_2 \in L^{\frac{1}{q_2}}([0,a),\mathbb{R}^+)$ such that $\alpha(f(t,B)) \le m_2(t)\alpha(B)$ for almost all $t \in [0,a)$ and B a bounded set in $\mathcal{C}$.

Lemma [6]

Assume that the hypotheses (H1) and (H2) hold. $x \in C([r,T],X)$ is a solution for fractional IVP (2.1) on $[r,T]$ for $T \in (0,a)$ if and only if x satisfies the following relation :

$$\begin{cases} x(\theta) = \varphi(\theta), & \text{for } \theta \in [-r,0], \\ x(t) = \varphi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_s)\, ds, & \text{for } t \in [0,T]. \end{cases}$$

Proof (see [6])

Theorem*

Assume that hypotheses (H1)-(H3) hold. Then, for every $\phi \in C$, there exists a solution $x \in C([r,T],X)$ for fractional IVP (1.1) with some $T \in (0,a)$.

Corollary

Assume that hypotheses (H1)-(H3) hold. Then, for every $\phi \in C$, there exist $T \in (0,a)$ and a sequence of continuous function $x^n : [r,T] \longrightarrow X$, such that

1. $x^n(t)$ are absolutely continuous on $[0,T]$;
2. $x_0^n = \varphi$, for every $n \geq 1$.
3. extracting a subsequence which is labeled in the same way such that $x^n(t) \to x(t)$ uniformly on $[-r,T]$ and $x : [-r,T] \to X$ is a solution for fractional IVP (2.1).

Example

Consider the infinite system of fractional functional differential equations

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} x_n(t) = \frac{1}{nt^{1/3}} x_n^2(t-r), & \text{for } t \in (0,a), \\ x_n(\theta) = \varphi(\theta) = \frac{\theta}{n}, & \text{for } \theta \in [-r,0], n = 1,2,3,\ldots \end{cases} \tag{2.2}$$

Let $E = c_0 = \{x = (x_1, x_2, x_3, \ldots) : x_n \to 0\}$ with norm $|x| = \sup_{n \geq 1} |x_n|$.

Then the infinite system (2.2) can be regarded as a fractional IVP of form (2.1) in E. In this situation, $q = \frac{1}{2}, x = (x_1, \ldots, x_n, \ldots), x_t = x(t-r) = (x_1(t-r), \ldots, x_n(t-r), \ldots)$, $\varphi(\theta) = \left(\theta, \frac{\theta}{2}, \ldots, \frac{\theta}{n}, \ldots\right)$

for $\theta \in [-r, 0]$ and $f = (f_1, \ldots, f_n, \ldots)$, in which

$$f_n(t, x_t) = \frac{1}{nt^{1/3}} x_n^2(t-r) \tag{2.3}$$

It is obvious that conditions (H1) and (H2) are satisfied.

Now, we check the condition (H3) Let $t \in (0, a), R > 0$ be given and $\{w^{(m)}\}$ be any sequence in $f(t, B)$, where $w^{(m)} = \left(w_1^{(m)}, \ldots, w_n^{(m)}, \ldots\right)$ and $B = \{z \in \mathcal{C} : \|z\|_* \leq R\}$ is a bounded set in $\mathcal{C}$ By (2.3), we have

$$0 \leq w_n^{(m)} \leq \frac{R^2}{nt^{1/3}}, \quad n, m = 1, 2, 3, \ldots \tag{2.4}$$

So, $\left\{w_n^{(m)}\right\}$ is bounded and, by the diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$w_n^{(m_i)} \to w_n \text{ as } i \to \infty, \quad n = 1, 2, 3, \ldots$$

which implies by virtue of (2.4) that

$$0 \leq w_n \leq \frac{R^2}{nt^{1/3}}, \quad n = 1, 2, 3, \ldots$$

Hence $w = (w_1, \ldots, w_n, \ldots) \in c_0$. It is easy to see from (2.4) that

$$\left|w^{(m_i)} - w\right| = \sup_n \left|w_n^{(m_i)} - w_n\right| \to 0 \text{ as } i \to \infty$$

Thus, we have proved that $f(t, B)$ is relatively compact in c_0 for $t \in (0, a)$, which means that $f(t, B) = 0$ for almost all $t \in [0, a)$ and B a bounded set in $\mathcal{C}$. Hence, the condition (H3) is satisfied. Finally, from Theorem*, we can conclude that the infinite system (2.2) has a continuous solution.

2.2 Cauchy Problems via Topological Degree Method

We consider the following problem via a coincidence degree for condensing mapping in a Banach space X

$$\begin{cases} {}_0^C D_t^q u(t) = f(t, u(t)), \quad t \in J := [0, T] \\ \\ u(0) + g(u) = u_0 \end{cases} \tag{2.5}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $q \in (0, 1), u_0$ is an element of $X, f : J \times X \to X$ is continuous. The nonlocal term $g : C(J, X) \to X$ is a given function, here $C(J, X)$ is the Banach space of all continuous functions from J into X with the norm $\|u\| := \sup_{t \in J} |u(t)|$ for $u \in C(J, X)$.

2.2.1 Qualitative Analysis

This subsection deals with existence of solutions for the nonlocal problem (2.5).

Definition

A function $u \in C^1(J, X)$ is said to be a solution of the nonlocal problem (2.5) if u satisfies the equation ${}_0^C D_t^q u(t) = f(t, u(t))$ a.e. on J, and the condition $u(0) + g(u) = u_0$.

Lemma [6]

A function $u \in C(J, X)$ is a solution of the fractional integral equation

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s)) ds \tag{2.6}$$

if and only if u is a solution of the nonlocal problem (2.5). We make some following assumptions:

(H1) for arbitrary u, $v \in C(J, X)$, there exists a constant $K_g \in [0, 1)$ such that

$$|g(u) - g(v)| \leq K_g \|u - v\| \ ;$$

(H2) for arbitrary $u \in C(J, X)$, there exist $C_g, M_g > 0, q_1 \in [0, 1)$ such that

$$|g(u)| \leq C_g \|u\|^{q_1} + M_g;$$

(H3) for arbitrary $(t, u) \in J \times X$, there exist $C_f, M_f > 0, q_2 \in [0, 1)$ such that

$$|f(t, u)| \leq C_f |u|^{q_2} + M_f;$$

(H4) for any $r > 0$, there exists a constant $\beta_r > 0$ such that

$$\alpha(f(s, \mathcal{M})) \leq \beta_r \alpha(\mathcal{M})$$

for all $t \in J, \mathcal{M} \subset \mathfrak{B}_r := \{\|u\| \leq r : u \in C(J, X)\}$ and

$$\frac{2T^q \beta_r}{\Gamma(q+1)} < 1$$

Under the assumptions (H1)-(H4), we show that fractional integral equation (2.6) has at least one solution $u \in C(J, X)$. Define operators

$$F : C(J, X) \to C(J, X), \quad (Fu)(t) = u_0 - g(u), \quad t \in J$$

$$G : C(J, X) \to C(J, X), \quad (Gu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s)) ds, \quad t \in J$$

$$\mathrm{T} : C(J, X) \to C(J, X), \quad Tu = Fu + Gu$$

It is obvious that T is well defined. Then, fractional integral equation (2.6) can be written as the following operator equation

$$u = \mathrm{T}u = Fu + Gu$$

Thus, the existence of a solution for the nonlocal problem (2.5) is equivalent to the existence of a fixed point for operator T.

Lemma [6]

The operator $F : C(J, X) \to C(J, X)$ is Lipschitz with constant K_g. Consequently F is α -Lipschitz with the same constant K_g. Moreover, F satisfies the following growth condition:

$$\|Fu\| \leq |u_0| + C_g \|u\|^{q_1} + M_g$$

for every $u \in C(J, X)$.

Proof (see [6])

Lemma [6]

The operator $G : C(J, X) \to C(J, X)$ is continuous. Moreover, G satisfies the following growth condition:

$$\|Gu\| \leq \frac{T^q \left(C_f \|u\|^{q_2} + M_f\right)}{\Gamma(q+1)} \tag{2.7}$$

for every $u \in C(J, X)$.

Proof [6]

For that, let $\{u_n\}$ be a sequence of a bounded set $\mathfrak{B}_K \subseteq C(J, X)$ such that $u_n \to u$ in $\mathfrak{B}_K(K > 0)$. We have to show that $\|Gu_n - Gu\| \to 0$.

It is easy to see that $f\left(s, u_n(s)\right) \to f(s, u(s))$ as $n \to \infty$ due to the continuity of f. On the one hand, using (H3), we get for each $t \in J$ $(t-s)^{q-1} \mid f\left(s, u_n(s)\right) - f(s, u(s)) \mid \leq (t-s)^{q-1} 2\left(C_f K^{q_2} + M_f\right)$.

On the other hand, using the fact that the function $s \to (t-s)^{q-1} 2\left(C_f K^{q_2} + M_f\right)$ is integrable for $s \in [0, t], t \in J$, Lebesgue dominated convergence theorem yields

$\int_0^t (t-s)^{q-1} |f(s, u_n(s)) - f(s, u(s))| \, ds \to 0$ as $n \to \infty$. Then, for all $t \in J$,

$$|(Gu_n)(t) - (Gu)(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, u_n(s)) - f(s, u(s))| \, ds \to 0.$$

Therefore, $Gu_n \to Gu$ as $n \to \infty$ which implies that G is continuous. Relation (2.7) is a simple consequence of (H3).

Lemma [6]

The operator $G : C(J, X) \to C(J, X)$ is compact. Consequently G is α-Lipschitz with zero constant.

Proof (see [6])

Theorem [6]

Assume that (H1)-(H4) hold, then the nonlocal problem (2.6) has at least one solution $u \in C(J, X)$ and the set of the solutions of system (2.6) is bounded in $C(J, X)$.

Remark [6]

1. If the growth condition (H2) is formulated for $q_1 = 1$, then the conclusions of Theorem$^{'}$ remain valid provided that $C_g < 1$.

2. If the growth condition (H3) is formulated for $q_2 = 1$, then the conclusions of Theorem$^{'}$ remain valid provided that $\frac{T^q C_f}{\Gamma(q+1)} < 1$

3. If the growth conditions (H2) and (H3) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem$^{'}$ remain valid provided that $C_g + \frac{T^4 C_f}{\Gamma(q+1)} < 1$

Chapter 3

Fractional Differential Equations with Initial Conditions at Inner Points

3.1 Existence and Uniqueness Results

In this section, we study the initial value problem for nonlinear fractional differential equations with initial conditions at inner points. More precisely, we will prove a Peano type theorem of the fractional version. We begin with the definition of the solutions to this problem.

Consider initial value problem (IVP for short) (see [1] [2])

$$\begin{cases} {}^cD_a^\alpha y(x) = f(x, y(x)), \quad x \geq x_0 \in (a, b) \\ y(x_0) = y_0 \end{cases} \tag{3.1}$$

Where $0 < \alpha < 1$, ${}^cD_a^\alpha$, $f\,[a, b] \times X \longrightarrow X$ is a given.

Problem (3.1) is equivalent to the integral equation

$$y(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt. \tag{3.2}$$

We first give an existence result based on the Banach contraction principle.

Theorem[1] [1] [2]

Let $0 < \alpha < 1$, and $G = [a, b] \times X$. Let $f : G ß X$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L, i.e.

$$| f(x, y_2) - f(x, y_1) | \leq L \parallel y_2 - y_1 \parallel, (x, y_1), (x, y_2) \in G$$

Then for $(x_0, y_0) \in G$ with $x_0 < \alpha + \left(\dfrac{\Gamma(\alpha + 1)}{2L} \right)^{\frac{1}{\alpha}}$, there exists a unique solution $y \in C\left([a, x_0] \times X\right)$ to the IVP.

proof [1] [2]

We define a mapping $T : C\left([a, x_0], X\right) \to C\left([a, x_0], X\right)$ by

$$(Ty)(x) = y_0 + \int_a^x \frac{(x - t)^{\alpha - 1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^{x_0} \frac{(x_0 - t)^{\alpha - 1}}{\Gamma(\alpha)} f(t, y(t))dt$$

for $y \in C\left([a, x_0], X\right)$ and $x \in [a, x_0]$. Then for any $y_1, y_2 \in C\left([a, x_0], X\right)$ and $x \in [a, x_0]$, we have

$$\begin{aligned}
\left\|(Ty_2)(x) - (Ty_1)(x)\right\| \leq & \int_a^x \frac{(x - t)^{\alpha - 1}}{\Gamma(\alpha)} \left\| f\left(t, y_2(t)\right) - f\left(t, y_1(t)\right) \right\| dt \\
& + \int_a^{x_0} \frac{(x_0 - t)^{\alpha - 1}}{\Gamma(\alpha)} \left\| f\left(t, y_2(t)\right) - f\left(t, y_1(t)\right) \right\| dt \\
\leq & L \int_a^x \frac{(x - t)^{\alpha - 1}}{\Gamma(\alpha)} \left\| y_2(t) - y_1(t) \right\| dt \\
& + L \int_a^{x0} \frac{(x_0 - t)^{\alpha - 1}}{\Gamma(\alpha)} \left\| y_2(t) - y_1(t) \right\| dt \\
\leq & \frac{2L\left(x_0 - a\right)^{\alpha}}{\Gamma(\alpha + 1)} \left\| y_2 - y_1 \right\|_{\infty}
\end{aligned}$$

And hence

$$\left\| Ty_2 - Ty_1 \right\|_{\infty} \leq K \left\| y_2 - y_1 \right\|_{\infty}$$

with $K = \frac{2L(x_0 - a)^{\alpha}}{\Gamma(\alpha + 1)}$.

Since $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$, we get that $\frac{2L(x_0 - a)^{\alpha}}{\Gamma(\alpha+1)} < 1$. Thus an application of Banach's fixed point theorem yields the existence and uniqueness of solution to our integral equation (3.2).

Remark[1] [2]

The condition $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$ means that the point x_0 cannot be far away from a. However, the following example shows that we cannot expect that there exists a solution to 1.1 for each $x_0 \in (a, b]$.

Example [1] [2]

Consider the differential equation with the Caputo fractional derivative

$$^{c}D_0^{\frac{1}{2}} y(x) = \frac{2\sqrt{x}}{\sqrt{\pi c}} y^2(x)$$

Consider the differential equation with the Caputo fractional derivative

$$y(x) = \frac{1}{\sqrt{c-x}}$$

whose existence interval is $[0; c)$.

However, from the proof of Theorem[1] we can see that if the Lipschitz constant L is small enough, then x_0 can be extended to the whole interval. Thus we have the following result.

Thoerem [1] [2]

Let $0 < \alpha < 1$ and $G = [a; b] \times R$. Let $f : G \longrightarrow X$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L.
If $L < \dfrac{\Gamma(\alpha+1)}{2(a-b)^{\alpha}}$, then for every $(x_0; y_0) \in G$, there exists a unique solution $y \in C\,[a; x_0]$ to the IVP(3.1).

Now we prove an existence result to the initial value problem at inner points (3.1)based on the generalized Banach contraction principle.

Thoerem [1] [2]

Let $0 < \alpha < 1$ and $G = [a, b] \times X$. Let $f : G \to X$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L.

Suppose $(x_0, y_0) \in G$ with $x_0 \leq a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$. Then for $h > 0$, there exists a unique solution on $[a, x_0 + h]$ to IVP (3.1).

Proof [1]

A function $y \in C\left([a, x_0 + h]\,; X\right)$ is a solution to (1.1)if and only if y satisfies

$$y(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt \tag{3.3}$$

for $t \in [a, x_0 + h]$. By Theorem[1], there exists a unique function $y^* \in C\left([a, x_0]\,; X\right)$ satisfying

$$y^*(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f\left(t, y^*(t)\right) dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f\left(t, y^*(t)\right) dt \tag{3.4}$$

Extend y^* to $[a, x_0 + h]$, also denoted by y^*, by

$$\begin{cases} y^*(x) = y^*(x), & x \in [a, x_0] \\ y^*(x) = y_0, & x \in [x_0, x_0 + h]\,. \end{cases} \tag{3.5}$$

For $z \in C\left([x_0, x_0 + h]\,; X\right)$ with $z\left(x_0\right) = 0$, we extend z to $[a, x_0 + h]$, still denoted by $\tilde{z}$, by

$$\begin{cases} \tilde{z}(x) = 0, & x \in [a, x_0] \\ \tilde{z}(x) = z(x), & x \in [x_0, x_0 + h]\,. \end{cases} \tag{3.6}$$

It is easily seen that a function $y \in C\left([a, x_0 + h]\,; X\right)$ satisfies (3.3) if and only if there is a function $z \in C\left([x_0, x_0 + h]\,; X\right)$ with $z\left(x_0\right) = 0$ such that $y = y^* + \tilde{z}$ on $[a, x_0 + h]$.

Moreover, y and y^* agree on $[a, x_0]$ and for $x \in [x_0, x_0+h]$, we have

$$\tilde{z}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f\left(t, y^*(t) + \tilde{z}(t)\right) dt$$

$$-\frac{1}{\Gamma(\alpha)} \int_a^{x_0} (x_0-t)^{\alpha-1} f\left(t, y^*(t) + \tilde{z}(t)\right) dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f\left(t, y^*(t) + \tilde{z}(t)\right) dt + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f\left(t, y^*(t) + \tilde{z}(t)\right) dt.$$

Due to (3.5) and (3.6) this equation can be rewritten as

$$\begin{aligned} z(x) &= \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f\left(t, y^*(t)\right) dt + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f\left(t, z(t) + y_0\right) dt \\ &= g(x) + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f\left(t, z(t) + y_0\right) dt \end{aligned}$$

for $x \in [x_0, x_0+h]$, where $g(x) = \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f\left(t, y^*(t)\right) dt$ with $g(x_0) = 0$. Since y^* is uniquely determined on $[a, x_0], g$ is a known function. Let $W = \{z \in C([x_0, x_0+h]; \mathbb{R}^m) : z(x_0) = 0\}$ endowed with the supremum norm $\|z\|_\infty = \sup_{x \in [x_0, x_0+h]} \|z(x)\|$. Then W becomes a Banach space. Define an operator $T: W \to W$ by

$$(Tz)(x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f\left(t, z(t) + y_0\right) dt$$

for $z \in W$ and all $x \in [x_0, x_0+h]$. Obviously if z is a fixed point of T, then $y = y^* + \tilde{z}$ is a solution to (3.1) and vise versa.

Below we prove that T has a unique fixed point in W by the generalized Banach contraction principle.

We first note that T is well-defined due to the continuity of the function g and the fact that $g(x_0) = 0$. Next we prove that for any $z_1, z_2 \in W$,

$$\|T^n z_2 - T^n z_1\|_\infty \leq \frac{(Lh^\alpha)^n}{\Gamma(n\alpha+1)} \|z_2 - z_1\|_\infty$$

for every $n \in \mathbb{N}$. In fact, take arbitrary $z_1, z_2 \in W$. Then for every $x \in [x_0, x_0+h]$, we have

$$\begin{aligned}\|Tz_2(x) - Tz_1(x)\| &\leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \|f(t, z_2(t) + y_0) - f(t, z_1(t) + y_0)\| \, dt \\ &\leq \frac{L}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \|z_2(t) - z_1(t)\| \, dt \\ &= L I_{x0}^{\alpha} \|z_2(\cdot) - z_1(\cdot)\| (x)\end{aligned}$$

Then we have

$$\begin{aligned}\left\|T^2 z_2(x) - T^2 z_1(x)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \|f(t, Tz_2(t) + y_0) - f(t, Tz_1(t) + y_0)\| \, dt \\ &\leq \frac{L}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \|Tz_2(t) - Tz_1(t)\| \, dt \\ &= \frac{L^2}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \left(I_{x_0}^{\alpha} \|z_2 - z_1\| (t)\right) dt \\ &= L^2 I_{x_0}^{2\alpha} \|z_2(\cdot) - z_1(\cdot)\| (x).\end{aligned}$$

By induction, we deduce that for $n \in \mathbb{N}$ and every $x \in [x_0, x_0 + h]$,

$$\|T^n z_2(x) - T^n z_1(x)\| \leq L^n I_{x_0}^{n\alpha} \|z_2(\cdot) - z_1(\cdot)\| (x) = \frac{L^n}{\Gamma(n\alpha)} \int_{-\infty}^{x} (x-t)^{n\alpha-1} \|z_2(t) - z_1(t)\| \, dt$$

$$\begin{aligned}&\leq \tfrac{L^n}{\Gamma(n\alpha)} \textstyle\int_{x_0}^{x} (x-t)^{n\alpha-1} dt \, \|z_2 - z_1\|_{\infty} \\ &= \tfrac{L^n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \|z_2 - z_1\|_{\infty} \\ &\leq \tfrac{L^n}{\Gamma(n\alpha+1)} h^{n\alpha} \|z_2 - z_1\|_{\infty}\end{aligned}$$

Take supremum on both side we obtain that

$$\|T^n z_2 - T^n z_1\|_{\infty} \leq \frac{(Lh^{\alpha})^n}{\Gamma(n\alpha + 1)} \|z_2 - z_1\|_{\infty}$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{(Lh^{\alpha})^n}{\Gamma(n\alpha+1)} = 0$, we can take a natural number n_0 large enough such that. $\frac{(Lh^{\alpha})^{n_0}}{\Gamma(n_0\alpha+1)} < \frac{1}{2}$. Hence

$$\|T^{n_0} z_2 - T^{n_0} z_1\|_{\infty} \leq \frac{1}{2} \|z_2 - z_1\|_{\infty}$$

By the generalized Banach contraction principle, T has a unique fixed point z in W and this completes the proof.

Conclusion

In this dissertation we present some results of existence and uniqueness of the initial value problem for nonlinear fractional differential equations with initial conditions at inner points, we use the Caputo derivative and some fix point theorems.

Bibliography

[1] Qixiang Dong, Existence and viability for fractional differential equations with initial conditions at inner points, J. Nonlinear Sci. Appl. **9** (2016), 2590-2603.

[2] Xiaoping Xu, Guangxian Wu, Qixiang Dong, Fractional Differential Equations with Initial Conditions at Inner Points in Banach Spaces J Scientific research Publishing.

[3] lgor Podlubny, Mathematics In Science And Engineering, Fractional Differential equations, Technical University of Kosice, Slovak Republic, Vol 198.

[4] Constantin Milici, Gheorghe Draganescu, J. Tenreiro Machado, Introduction to Fractional Differential equations, Vol 25, Springer.

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, 204 (2006).

[6] Young Zhou, JinRong Wang, Lu Zhung, Basic theory of Fractional Differential equations, Vol 2, World Scientific.

[7] Hacen DIB, Equations différentielles fractionnaires, EDA-EDO (4ème Ecole) ,Tlemcen 23-27 mai 2009.

[8] Kei Diethlem, The Analysis of Fractional Differential Equation, Springer.

[9] R. Vilela Mendes, Introduction to fractional calculus.

[10] A.Erdelyi , W.Magnus, Oberhettinger F and Tricomi F, Higher Transcendental Functions, Vol.III,Krieger Pub, Melbourne, Florida, (1981).

[11] R. Vilela Mendes, Introduction to fractional calculus.