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Some Common Fixed Point Theorems in Complex Valued Metric Spaces

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Introduction

Fixed point theorems constitute an important and interesting aspect of applicable mathematics and provide solutions to several linear and nonlinear problems arising in biological, engineering and physical sciences.

The origins of metric contraction principles and, ergo, metric fixed point theory itself, rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations.

This method is associated with the names of such celebrated nineteenth century mathematicians as Cauchy, Liouville, Lipschitz, Peano, and, especially, Picard.

The origin of fixed point theory is a method of successive approximations used to prove the existence of differential equation solutions introduced by Picard in 1890.

However it is the Polish mathematician **Stefan Banach**, in his thesis (1922), who is credited with placing the ideas underlying the method into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equations. We can distinguish three major approaches in fixed point theory:

1. metric approach.
2. topological approach
3. discrete approach

Historically, these approaches were initiated by the discovery of three major theorems:

1. Banach fixed point theorem.

2. Brouwer fixed point theorem.

3. Tarski fixed point theorem.

In this thesis, we are concerned with the first approach.

Metric fixed point theory is an important mathematical discipline because of its applications in different areas such as variational and linear inequalities, optimization theory, boundary value problems, etc.

In the context of metric fixed point theory many researchers have been working on generalizing the Banach fixed point theorem by either:

1) Generalizing the type of the contraction

In the literature, there are plenty of extensions of the famous Banach contraction principle such as:

- nonexpansive mapping
- F-contraction
- Meir-Keeler contraction
- Suzuki-contraction

2) by extending the metric space itself:

- b-metric spaces
- p-metric spaces
- D-metric spaces
- G-metric spaces

- rectangular metric space
- quasimetric spaces
- Probabilistic metric spaces, etc...

and by now there exists considerable literature on all these generalizations of metric spaces. For more details, you can see for exemple [**?**]

Recently, Azam and al.[1] was the first introduced the **complex valued metric spaces** which is more general than well-know metric spaces and also gave common fixed point theorems for mappings satisfying generalized contraction condition.

The aim of this thesis is to present a class of some recent advances in this theory, that is, Fixed Point Theorems in **Complex Valued Metric Spaces**.

In first chapter we presente some preliminaire and the original Banach fixed point theorem.

The second chapter introduce the complex valued metric spaces and related fixed point theorems. It is the the detail of the article [1].

The third chapter give more results in complex valued metric spaces and related fixed point theorems. It is the the detail of the article [17].

Finaly the fourth chapter presente a generalized common fixed point theorems in complex valued metric spaces and the study of an important application. it is the detail of article [19].

Chapter 1

Metric Spaces and Banach Contraction Mapping Theorem

In this chapter, we will mention for a metric spaces and some of their properties, and giving some examples. Finally We will also talk about Banach's fixed point theorem and its proof.

1.1 Metric Spaces

Definition 1.1.1. *Let X be a non-empty set. A collection of J of subsets of X is called a topology on X, if*

1. $\phi, X \in \mathcal{T}$
2. $G_1 \cap G_2 \in \mathcal{T}$ *for* $G_1, G_2 \in \mathcal{T}$
3. $\bigcup_{G \in \mathcal{F}} G \in \mathcal{T}$ *for any* $\mathcal{F} \subseteq \mathcal{T}$.

Any subset of X belonging to $\mathcal{T}$ is called an open set or more precisely $\mathcal{T}$-open set. The pair $(X, \mathcal{T})$ is called a topological space. Given a topological space $(X, \mathcal{T})$,

Definition 1.1.2. *A topological space X is said to be Hausdorff if given any two points $x, y \in X$ there are open sets U and V in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$*

Definition 1.1.3. *Let X be a non-empty set. A metric on X, or distance function, associates to each pair of elements $x, y \in X$ a real number $d(x, y)$ such that*

1. $d(x, y) = 0 \Longleftrightarrow x = y$

2. $d(x, y) = d(y, x)$

3. $d(x, z) \leq d(x, y) + d(y, z)$

Exemple 1. *$X = \mathbb{R}$, The standard metric is given by $d(x, y) = |x - y|$. There are many other metrics on $\mathbb{R}$, for example:*

$$d(x, y) = |e^x - e^y|$$

Definition 1.1.4. *Let (X, d) be a metric space, let $x \in X$ and let $r > 0$. The open ball centered at x, with radius r, is the set*

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

and the closed ball is the set

$$B[x, r] = \{y \in X : d(x, y) \leq r\}$$

Note that in $\mathbb{R}$ with the usual metric the open ball is $B(x, r) =]x - r, x + r[$, an open interval, and the closed ball is $B[x, r] = [x - r, x + r]$, a closed interval.

Definition 1.1.5. *A subset U of a metric space (X, d) is said to be open, if for each point $x \in U$ there is an $r > 0$ such that the open ball $B(x, r)$ is contained in U. Clearly X itself is an open set, and by convention the empty set $\emptyset$ is also considered to be open.*

Proposition 1.1.1. *Every open ball $B(x, r)$ is an open set.*

Proof. For if $y \in B(x,r)$, choose $\delta = r - d(x,y)$. We claim that $B(y,\delta) \subset B(x,r)$. If $z \in B(y,\delta)$, *i.e.*, $d(z,y) < \delta$, then by the triangle inequality

$d(z,x) \leq d(z,y) + d(y,x) < \delta + d(x,y) = r$. So $z \in B(x,r)$. □

Definition 1.1.6. *A subset F of (X,d) is said to be closed, if its complement F^c is open.*

Note that closed does not mean not open. In a metric space the sets $\emptyset$ and X are both open and closed. In $\mathbb{R}$ we have: $]a,b[$ is open. $[a,b]$ is closed, since its complement $]-\infty, a[\cup]b, +\infty[$ is open. $[a,b[$ is not open, since there is no open ball $B(a,r)$ contained in the set. Nor is it closed, since its complement $]-\infty, a[\cup[b, +\infty[$ isnt open (no ball centred at b can be contained in the set).

Remark 1.1.1. *Every metric space is a Hausdorff topological space.*

Definition 1.1.7. *We say $x_n \longrightarrow x$ (i.e. (x_n) tends to x or converges to x) if $d(x_n,x) \longrightarrow 0$ as $n \longrightarrow +\infty$. That is, for all $\epsilon > 0$ there is an N such that $d(x_n,x) < \epsilon$ for $n \geq N$.*

Definition 1.1.8. *A sequence (x_n) in a metric space (X,d) is a Cauchy sequence if for any $\epsilon > 0$ there is an N such that $d(x_n,x_m) < \epsilon$ for all $n,m \geq N$*

Exemple 2. *take $x_n = \dfrac{1}{n}$ in $\mathbb{R}$ with the usual metric. Now $d(x_n,x_m) = \left|\dfrac{1}{n} - \dfrac{1}{m}\right|$ Suppose that n and m are both at least as big as N, then $d(x_n,x_m) \leq \dfrac{1}{N}$.*

Proposition 1.1.2. *Suppose that (x_n) is a convergent sequence in a metric space (X,d), i.e, there is a limit point x such that $d(x_n,x) \longrightarrow 0$. Then (x_n) is a Cauchy sequence.*

Proof. take $\epsilon > 0$. Then there is an N such that $d(x_n,x) < \dfrac{\epsilon}{2}$ whenever $n \geq N$. Now suppose both $n \geq N$ and $m \geq N$. Then $d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) = d(x_n,x) + d(x_m,x) < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$ and we are done. □

Proposition 1.1.3. *Every subsequence of a Cauchy sequence is a Cauchy sequence.*

Proof. if (x_n) is Cauchy and (x_{n_k}) is a subsequence, then given $\epsilon > 0$ there is an N such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$. Now there is a K such that $n_k \geq N$ whenever $k \geq K$. So $d(x_{n_k}, x_{n_l}) < \epsilon$ whenever $k, l \geq K$. □

Definition 1.1.9. *A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit in X.*

1.2 Banach's Contraction Mapping Theorem

Definition 1.2.1. *Let (X, d) be a metric space. A map $\Phi : X \longrightarrow X$ is a contraction mapping, if there exists a constant $k < 1$ such that $d(\Phi(x), \Phi(y)) \leq kd(x, y)$ for all $x, y \in X$.*

Exemple 3. *Take $X = [0, 1]$, usual metric, and $\Phi(x) = \dfrac{x^2}{3}$ Then*

$$d\left(\Phi(x), \Phi(y)\right) = \left|\frac{x^2}{3} - \frac{y^2}{3}\right| = \left|\frac{1}{3}(x-y)(x+y)\right| \leq \frac{2}{3}\left|x - y\right| = \frac{2}{3}d(x, y)$$

So Φ is a contraction mapping, with $k = \dfrac{2}{3}$.

Theorem 1.2.1 (Banach's Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space, and let $\Phi : X \longrightarrow X$ be a contraction mapping. Then Φ has a unique fixed point.*

Proof. Notice first that if $x_1, x_2 \in X$ are fixed points of f, then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$$

which imply $x_1 = x_2$. Choose now any $x_0 \in X$, and define the iterate sequence $x_{n+1} = f(x_n)$. By induction on n,

$$d(x_{n+1}, x_n) \leq \lambda^n d(f(x_0), x_0)$$

. If $n \in \mathbb{N}$ and $m \geq 1$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (\lambda^{n+m} + \cdots + \lambda^n) d(f(x_0), x_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0). \end{aligned} \tag{1}$$

Hence (x_n) is a Cauchy sequence, and admits a limit $\bar{x} \in X$, for X is complete. Since f is continuous, we have $f(\bar{x}) = \lim\limits_{n \longrightarrow +\infty} f(x_n) = \lim\limits_{n \longrightarrow +\infty} x_{n+1} = \bar{x}$. □

Remark 1.2.1. *Notice that letting $m \longrightarrow +\infty$ in (1) we find the relation*

$$d(x_n, \bar{x}) \leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0)$$

which provides a control on the convergence rate of (x_n) to the fixed point $\bar{x}$. The completeness of X plays here a crucial role. Indeed, contractions on incomplete metric spaces may fail to have fixed points.

Exemple 4. *Let $X =]0, 1]$ with the usual distance. Define $f : X \longrightarrow X$ as $f(x) = \dfrac{x}{2}$. is clear that f a contraction mapping but it's have not any fixed point .*

Corollary 1.2.1. *Let X be a complete metric space and let $f : X \to X$. If f^n is a contraction, for some $n \geq 1$, then f has a unique fixed point $\bar{x} \in X$.*

Proof. Let $\bar{x}$ be the unique fixed point of f^n, given by Theorem (1.2.1) Then

$f^n(f(\bar{x})) = f(f^n(\bar{x})) = f(\bar{x})$, which implies $f(\bar{x}) = \bar{x}$. Since a fixed point of f is clearly a fixed point of f^n, we have uniqueness as well □

Chapter 2

First Common Fixed Point Theorem in Complex Valued Metric Spaces

In this chapter we will define a partial order on $\mathbb{C}$, and see a definition of a complex valued metric, and we will talk about the topology defined by this new metric, and we will see the principal theory in this chapter, which is an extension of Banach's fixed point theorem, and giving some examples.

This chapter is the the detail of the article [1].

2.1 Partial Order in $\mathbb{C}$ and New Definitions

Definition 2.1.1. *[1] Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$ Define a partial order $\precsim$ on $\mathbb{C}$ as follows:*

$$z_1 \precsim z_2 \Longleftrightarrow (Re(z_1) \leq Re(z_2)) \wedge (Im(z_1) \leq Im(z_2))$$

In particular, we will write $z_1 \precnsim z_2$ if $(z_1 \neq z_2) \wedge (z_1 \precsim z_2)$ and we will write $z_1 \prec z_2$ if $(Re(z_1) < Re(z_2)) \wedge (Im(z_1) < Im(z_2))$.

Note that:

$$0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$$

$$(z_1 \precsim z_2) \wedge (z_2 \prec z_3) \Longrightarrow z_1 \prec z_3$$

Definition 2.1.2. *[1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \longrightarrow \mathbb{C}$, satisfies:*

1. *for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.*

2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*

3. *$d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$*

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$

$$B(x, r) \cap (A - \{x\}) \neq \emptyset$$

A is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

Remark 2.1.1. *(X, d) is a hausdorff topological space.*

Definition 2.1.3. *[1] Let (x_n) be a sequence in X and $x \in X$. If for every $0 \prec c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \prec c$, then (x_n) is said to be convergent, (x_n) converges to x and x is the limit point of (x_n). We denote this by $\lim_{n \longrightarrow +\infty} x_n = x$, or $x_n \longrightarrow x$, as $n \longrightarrow +\infty$.*

Definition 2.1.4. *[1] If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, and all $m \in \mathbb{N}$: $d(x_n, x_{n+m}) \prec c \in \mathbb{C}$, then (x_n) is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space.*

Lemma 2.1.1. *[1] Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Proof. Suppose that (x_n) converges to x. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that

$$d(x_n, x) \prec c \text{ for all } n > N$$

Therefore,

$$|d(x_n, x)| < |c| = \epsilon \quad \text{ for all } n > N$$

It follows that

$$|d(x_n, x)| \to 0 \text{ as } n \to \infty$$

Conversely, suppose that $|d(x_n, x)| \to 0$ as $n \to \infty$. Then given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Longrightarrow z \prec c$$

For this δ, there is a natural number N such that

$$|d(x_n, x)| < \delta \quad \text{ for all } n > N$$

This means that $d(x_n, x) \prec c$ for all $n > N$. Hence (x_n) converges to x. □

Lemma 2.1.2. *[1] Let (X, d) be a complex valued metric space and let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Proof. Suppose that (x_n) is a Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that:

$$d\left(x_n, x_{n+m}\right) \prec c \quad \text{for all } n > N$$

Therefore,

$$\left|d\left(x_n, x_{n+m}\right)\right| \prec |c| = \epsilon \quad \text{for all } n > N$$

It follows that

$$\left|d\left(x_n, x_{n+m}\right)\right| \to 0 \text{ as } n \to \infty$$

Conversely, suppose that $\left|d\left(x_n, x_{n+m}\right)\right| \to 0$ as $n \to \infty$. For given $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Longrightarrow z \prec c$$

For this δ, there is a natural number N such that:

$$\left|d\left(x_n, x_{n+m}\right)\right| < \delta \text{ for all } n > N.$$

That is $d\left(x_n, x_{n+m}\right) \prec c$ for all $n > N$ and so (x_n) is a Cauchy sequence. □

2.2 An Extension Of The Banach Fixed Point Theorem

Theorem 2.2.1. *[1] Let (X, d) be a complete complex valued metric space and let the mappings $S, T : X \longrightarrow X$ satisfy:*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then S, T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k}$$

$$x_{2k+2} = Tx_{2k+1}, \quad k = 0, 1, 2, \ldots$$

Then,

$$\begin{aligned}
|d(x_{2k+1}, x_{2k+2})| &= |d(Sx_{2k}, Tx_{2k+1})| \\
&\leq \lambda |d(x_{2k}, x_{2k+1})| + \frac{\mu |d(x_{2k+1}, Tx_{2k+1})||d(x_{2k}, Sx_{2k})|}{|1 + d(x_{2k}, x_{2k+1})|} \\
&\leq \lambda |d(x_{2k}, x_{2k+1})| + \frac{\mu |d(x_{2k+1}, x_{2k+2})||d(x_{2k}, x_{2k+1})|}{|1 + d(x_{2k}, x_{2k+1})|} \\
&\leq \lambda |d(x_{2k}, x_{2k+1})| + \mu |d(x_{2k+1}, x_{2k+2})|
\end{aligned}$$

since $|d(x_{2k}, x_{2k+1})| \leq |1 + d(x_{2k}, x_{2k+1})|$

$$\leq \frac{\lambda}{1-\mu} |d(x_{2k}, x_{2k+1})|.$$

Similarly,

$$\begin{aligned}
|d(x_{2k+2}, x_{2k+3})| &= |d(Sx_{2k+2}, Tx_{2k+1})| \\
&\leq \lambda |d(x_{2k+2}, x_{2k+1})| + \frac{\mu |d(x_{2k+1}, Tx_{2k+1})||d(x_{2k+2}, Sx_{2k+2})|}{|1 + d(x_{2k+2}, x_{2k+1})|} \\
&\leq \lambda |d(x_{2k+2}, x_{2k+1})| + \frac{\mu |d(x_{2k+1}, x_{2k+2})||d(x_{2k+2}, x_{2k+3})|}{|1 + d(x_{2k+1}, x_{2k+2})|} \\
&\leq \lambda |d(x_{2k+2}, x_{2k+1})| + \mu |d(x_{2k+2}, x_{2k+3})| \\
&\leq \frac{\lambda}{1-\mu} |d(x_{2k+2}, x_{2k+1})|.
\end{aligned}$$

Now with $h = \dfrac{\lambda}{1-\mu}$, we have

$$\begin{aligned}
|d(x_{n+1}, x_{n+2})| &\leq h |d(x_n, x_{n+1})| \\
&\leq \cdots \leq h^{n+1} |d(x_0, x_1)|.
\end{aligned}$$

So for any $m > n$,

$$\begin{aligned}
|d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\
&\leq \left[h^n + h^{n+1} + \cdots + h^{m-1}\right] |d(x_0, x_1)| \\
&\leq \left[\frac{h^n}{1-h}\right] |d(x_0, x_1)|
\end{aligned}$$

and so

$$|d(x_{\mathrm{m}}, x_n)| \leq \frac{h^n}{1-h} |d(x_0, x_1)| \to 0, \qquad \text{as } m, n \to \infty.$$

This implies that (x_n) is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. It follows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned}
|z| &\leq |d(u, x_{2k+2})| + |d(x_{2k+2}, Su)| \\
&\leq |d(u, x_{2k+2})| + |d(Tx_{2k+1}, Su)| \\
&\leq |d(u, x_{2k+2})| + \lambda|d(x_{2k+1}, u)| + \frac{\mu|d(x_{2k+1}, Tx_{2h+1})||d(u, Su)|}{|1 + d(u, x_{2k+1})|} \\
&\leq |d(u, x_{2k+2})| + \lambda|d(x_{2k+1}, u)| + \frac{\mu|d(x_{2k+1}, x_{2k+2})||z|}{|1 + d(u, x_{2k+1})|}
\end{aligned}$$

This implies that

$$|z| \leq |d(u, x_{2k+2})| + \lambda|d(x_{2k+1}, u)| + \frac{\mu|d(x_{2k+1}, x_{2k+2})||z|}{|1 + d(u, x_{2k+1})|}.$$

That is $|z| = 0$, a contradiction and, hence, $u = Su$. It follows similarly that $u = Tu$. We now show that S and T have unique common fixed point. For this, assume that u^* in X is a second common fixed point of S and T. Then

$$\begin{aligned}
d(u, u^*) &= d(Su, Tu^*) \\
&\precsim \lambda d(u, u^*) + \frac{\mu d(u, Su) d(u^*, Tu^*)}{1 + d(u, u^*)} \\
&\precsim \lambda d(u, u^*).
\end{aligned}$$

This implies that $u^* = u$, completing the proof of the theorem. □

Corollary 2.2.1. *[1] Let (X, d) be a complete complex valued metric space and let the mapping $T : X \longrightarrow X$ satisfy:*

$$d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Tx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Corollary 2.2.2. *[1] Let (X,d) be a complete complex valued metric space and $T : X \longrightarrow X$ satisfy:*

$$d(T^n x, T^n y) \precsim \lambda d(x,y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x,y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof. By Corollary (2.2.1) we obtain a unique $v \in X$ such that

$$T^n v = v$$

The result then follows from the fact that

$$\begin{aligned} d(Tv, v) &= d\left(TT^n v, T^n v\right) = d\left(T^n T v, T^n v\right) \\ &\precsim \lambda d(Tv, v) + \frac{\mu d\left(Tv, T^n T v\right) d\left(v, T^n v\right)}{1 + d(Tv, v)} \\ &\precsim \lambda d(Tv, v) + \frac{\mu d\left(Tv, TT^n v\right) d(v, v)}{1 + d(Tv, v)} = \lambda d(Tv, v) \end{aligned}$$

So $Tv = v$, This implies v is a fixed point of T.

We now show that T have unique fixed point. For this, assume that $v_1 \in X$ is a second fixed point of T

$$Tv_1 = v_1 \Rightarrow T^2 v_1 = Tv_1 = v_1 \Rightarrow \cdots \Rightarrow T^n v_1 = v_1$$

So v_1 is a fixed point of T^n, but T^n have a unique fixed point v. This implies that $v = v_1$

So T have a unique fixed point.

□

Exemple 5. *Let*

$$X_1 = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1, \operatorname{Im}(z) = 0\}$$

$$X_2 = \{z \in \mathbb{C} : 0 \leq \operatorname{Im}(z) \leq 1, \operatorname{Re}(z) = 0\}$$

and let $X = X_1 \cup X_2$. Then with $z = x + iy$, define

$$Tz = \begin{cases} ix & \text{if } z \in X_1 \\ \\ \frac{1}{2} y & \text{if } z \in X_2. \end{cases}$$

If d_u is usual metric on X then T is not contractive as

$$d_u\left(Tz_1, Tz_2\right) = \left|x_1 - x_2\right| = d_u\left(z_1, z_2\right) \qquad \text{if } z_1, z_2 \in X_1$$

Therefore, the Banach contraction theorem is not valid to find the unique fixed point 0 of T.

To apply the corollary(2.2.1), consider a complex valued metric $d : X \times X \longrightarrow \mathbb{C}$ as follows:

$$d\left(z_1, z_2\right) = \begin{cases} \dfrac{2}{3}\left|x_1 - x_2\right| + \dfrac{i}{2}\left|x_1 - x_2\right|, & \text{if } z_1, z_2 \in X_1, \\ \\ \dfrac{1}{2}\left|y_1 - y_2\right| + \dfrac{i}{3}\left|y_1 - y_2\right|, & \text{if } z_1, z_2 \in X_2, \\ \\ \left(\dfrac{2}{3}x_1 + \dfrac{1}{2}y_2\right) + i\left(\dfrac{1}{2}x_1 + \dfrac{1}{3}y_2\right), & \text{if } z_1 \in X_1, \quad z_2 \in X_2 \\ \\ \left(\dfrac{1}{2}y_1 + \dfrac{2}{3}x_2\right) + i\left(\dfrac{1}{3}y_1 + \dfrac{1}{2}x_2\right), & \text{if } z_1 \in X_2, z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1, z_1 = x_2 + iy_2 \in X$. Then (X, d) is a complete complex valued metric space and

$$d\left(Tz_1, Tz_2\right) \precsim \frac{3}{4} d\left(z_1, z_2\right) \qquad \text{for all } z_1, z_2 \in X$$

Chapter 3

More Common Fixed Point Theorems in Complex Valued Metric Spaces

In this chapter, we will talk about other common fixed point theorem in complex valued metric spaces, which are considered a generalization of the second chapter theorems, and giving some examples to show their importance.

This chapter is the the detail of the article [17].

Definition 3.0.1. *[17] Two families of self-mappings* $\{T_i\}_{i=1}^m$ *and* $\{S_i\}_{i=1}^n$ *are said to be pairwise commuting if:*

1. $T_iT_j = T_jT_i, i, j \in \{1, 2, \dots, m\}$.
2. $S_iS_j = S_jS_i, i, j \in \{1, 2, \dots, n\}$.
3. $T_iS_j = S_jT_i, i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$

3.1 Common Fixed Point Theorems

Theorem 3.1.1. *[17] If S and T are self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx)d(y, Ty) + \gamma d(y, Sx)d(x, Ty)}{1 + d(x, y)} \tag{3.1}$$

for all $x, y \in X$ where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, \ldots$ Then

$$\begin{aligned} d\left(x_{2k+1}, x_{2k+2}\right) =& d\left(Sx_{2k}, Tx_{2k+1}\right) \precsim \lambda d\left(x_{2k}, x_{2k+1}\right) \\ &+ \frac{\mu d\left(x_{2k}, Sx_{2k}\right) d\left(x_{2k+1}, Tx_{2k+1}\right) + \gamma d\left(x_{2k}, Tx_{2k+1}\right) d\left(x_{2k+1}, Sx_{2k}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)} \end{aligned}$$

Since

$$x_{2k+1} = Sx_{2k} \Rightarrow d\left(x_{2k+1}, Sx_{2k}\right) = 0$$

therefore

$$d\left(x_{2k+1}, x_{2k+2}\right) \precsim \lambda d\left(x_{2k}, x_{2k+1}\right) + \frac{\mu d\left(x_{2k}, x_{2k+1}\right) \cdot d\left(x_{2k+1}, x_{2k+2}\right)}{1 + d\left(x_{2k}, x_{2k+1}\right)}$$

so that

$$\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \leq \lambda \left|d\left(x_{2k}, x_{2k+1}\right)\right| + \frac{\mu \left|d\left(x_{2k}, x_{2k+1}\right)\right| \cdot \left|d\left(x_{2k+1}, x_{2k+2}\right)\right|}{\left|1 + d\left(x_{2k}, x_{2k+1}\right)\right|}$$

Since

$$\left|1 + d\left(x_{2k}, x_{2k+1}\right)\right| > \left|d\left(x_{2k}, x_{2k+1}\right)\right|$$

therefore

$$\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \leq \lambda \left|d\left(x_{2k}, x_{2k+1}\right)\right| + \mu \left|d\left(x_{2k+1}, x_{2k+2}\right)\right|$$

so that

$$\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \leq \frac{\lambda}{1 - \mu} \left|d\left(x_{2k}, x_{2k+1}\right)\right|.$$

Also,

$$\begin{aligned} d\left(x_{2k+2}, x_{2k+3}\right) =& d\left(T x_{2k+1}, S x_{2k+2}\right)=d\left(S x_{2k+2}, T x_{2k+1}\right) \precsim \lambda d\left(x_{2k+2}, x_{2k+1}\right) \\ & + \frac{\mu d\left(x_{2k+2}, S x_{2k+2}\right) d\left(x_{2k+1}, T x_{2k+1}\right)+\gamma d\left(x_{2k+1}, S x_{2k+2}\right) d\left(x_{2k+2}, T x_{2k+1}\right)}{1+d\left(x_{2k+2}, x_{2k+1}\right)}. \end{aligned}$$

Since

$$x_{2k+2}=T x_{2k+1} \Rightarrow d\left(x_{2k+2}, T x_{2k+1}\right)=0$$

therefore

$$d\left(x_{2k+2}, x_{2k+3}\right) \precsim \lambda d\left(x_{2k+2}, x_{2k+1}\right)+\frac{\mu d\left(x_{2k+2}, S x_{2k+2}\right) d\left(x_{2k+1}, T x_{2k+1}\right)}{1+d\left(x_{2k+2}, x_{2k+1}\right)}$$

so that

$$\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| \leq \lambda\left|d\left(x_{2k+2}, x_{2k+1}\right)\right|+\frac{\mu\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| \cdot\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|}{\left|1+d\left(x_{2k+2}, x_{2k+1}\right)\right|}.$$

As

$$\left|1+d\left(x_{2k+2}, x_{2k+1}\right)\right|>\left|d\left(x_{2k+2}, x_{2k+1}\right)\right|$$

therefore

$$\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| \leq \frac{\lambda}{1-\mu}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|.$$

Putting $h=\dfrac{\lambda}{1-\mu}$, we have (for all $n \in \mathbb{N}$)

$$\left|d\left(x_{n}, x_{n+1}\right)\right| \leq h\left|d\left(x_{n-1}, x_{n}\right)\right| \leq h^{2}\left|d\left(x_{n-2}, x_{n-1}\right)\right| \leq \cdots \leq h^{n}\left|d\left(x_{0}, x_{1}\right)\right|.$$

Therefore, for any $m>n$, we have

$$\begin{aligned} \left|d\left(x_{n}, x_{m}\right)\right| & \leq\left|d\left(x_{n}, x_{n+1}\right)\right|+\left|d\left(x_{n+1}, x_{n+2}\right)\right|+\left|d\left(x_{n+2}, x_{n+3}\right)\right|+\cdots+\left|d\left(x_{m-1}, x_{m}\right)\right| \\ & \leq\left[h^{n}+h^{n+1}+h^{n+2}+\cdots+h^{m-1}\right]\left|d\left(x_{0}, x_{1}\right)\right| \\ & \leq\left[\frac{h^{n}}{1-h}\right]\left|d\left(x_{0}, x_{1}\right)\right| \end{aligned}$$

so that

$$\left|d\left(x_{n}, x_{m}\right)\right| \leq\left[\frac{h^{n}}{1-h}\right]\left|d\left(x_{0}, x_{1}\right)\right| \rightarrow 0 \quad \text{ as } n \rightarrow \infty.$$

The sequence (x_n) is Cauchy. Since X is complete, there exists some $l \in X$ such that $x_n \to l$ as $n \to \infty$. On the contrary, let $l \neq Sl$ so that $d(l, Sl) = z > 0$ and henceforth we can have

$$\begin{aligned} z = d(l, Sl) &\precsim d\left(l, Tx_{2k+1}\right) + d\left(Tx_{2k+1}, Sl\right) \\ &\precsim d\left(l, x_{2k+2}\right) + \lambda d\left(l, x_{2k+1}\right) + \frac{\mu d(l, Sl) d\left(x_{2k+1}, Tx_{2k+1}\right) + \gamma d\left(x_{2k+1}, Sl\right) d\left(l, Tx_{2k+1}\right)}{1 + d\left(l, x_{2k+1}\right)} \\ &\precsim d\left(l, x_{2k+2}\right) + \lambda d\left(l, x_{2k+1}\right) + \frac{\mu z \cdot d\left(x_{2k+1}, Tx_{2k+1}\right) + \gamma d\left(x_{2k+1}, Sl\right) d\left(l, Tx_{2k+1}\right)}{1 + d\left(l, x_{2k+1}\right)} \end{aligned}$$

Also, for every k, we can write

$$|d(l, Sl)| \leq \left|d\left(l, x_{2k+2}\right)\right| + \lambda \left|d\left(l, x_{2k+1}\right)\right| + \frac{\mu |z| \cdot \left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \gamma \left|d\left(x_{2k+1}, Sl\right)\right| \cdot \left|d\left(l, x_{2k+2}\right)\right|}{\left|1 + d\left(l, x_{2k+1}\right)\right|}.$$

Making $k \to \infty$, one gets $|d(l, Sl)| = 0$ which is a contradiction so that $l = Sl$. Similarly, one can also show that $l = Tl$. To prove the uniqueness of common fixed point, let l^* (in X) be another common fixed point of S and T i.e. $l^* = Sl^* = Tl^*$. Then

$$\begin{aligned} d\left(l, l^*\right) = d\left(Sl, Tl^*\right) &\precsim \lambda d\left(l, l^*\right) + \frac{\mu d(l, Sl) d\left(l^*, Tl^*\right) + \gamma d\left(I^*, Sl\right) d\left(l, Tl^*\right)}{1 + d\left(l, l^*\right)} \\ &= \lambda d\left(l, l^*\right) + \frac{\gamma d\left(l^*, l\right) d\left(l, l^*\right)}{1 + d\left(l, l^*\right)} \end{aligned}$$

so that

$$\left|d\left(l, l^*\right)\right| \leq \lambda \left|d\left(l, l^*\right)\right| + \frac{\gamma \left|d\left(l^*, l\right)\right| \cdot \left|d\left(l, l^*\right)\right|}{\left|1 + d\left(l, l^*\right)\right|}.$$

Since

$$\left|1 + d\left(l, l^*\right)\right| > \left|d\left(l, l^*\right)\right|$$

therefore

$$\left|d\left(l, l^*\right)\right| \leq (\lambda + \gamma) \left|d\left(l, l^*\right)\right|$$

which is a contradiction so that $l = l^*$ (as $\lambda + \gamma < 1$). This completes the proof of the theorem.

□

Remark 3.1.1. *By setting $S = T$ in Theorem* (3.1.1)*, one deduces the following:*

Corollary 3.1.1. *[17] If $T : X \to X$ is a self-mapping defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(Tx,Ty) \precsim \lambda d(x,y) + \frac{\mu d(x,Tx)d(y,Ty) + \gamma d(y,Tx)d(x,Ty)}{1+d(x,y)}$$

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then T has a unique fixed point.

Theorem 3.1.2. *[17] If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X,d) such that the mappings S and T (with $T = T_1T_2\cdots T_m$ and $S = S_1S_2\cdots S_n$) satisfy condition (3.1), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. In view of Theorem (3.1.1), one can infer that T and S have a unique common fixed point l i.e. $Tl = Sl = l$. Now we are required to show that l is a common fixed point of all the component maps of both the families. In view of pairwise commutativity of the families $\{T_i\}_1^m$ and $\{S_i\}_1^n$, (for every $1 \leq k \leq m$) we can write

$$T_kl = T_kSl = ST_kl \text{ and } T_kl = T_kTl = TT_kl$$

which show that $T_kl($ for every $k)$ is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_kl = l$ (for every k) which shows that l is a common fixed point of the family $\{T_i\}_1^m$. Using the foregoing arguments, one can also show that (for every $1 \leq k \leq n)S_kl = l$. This completes the proof of the theorem. □

By setting $T_1 = T_2 = \cdots = T_m = F$ and $S_1 = S_2 = \cdots = S_n = G$, in Theorem (3.1.2), we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.1.2. *[17] If F and G are two commuting self-mappings defined on a complete complex valued metric space (X,d) satisfying the condition*

$$d\left(F^mx, G^ny\right) \precsim \lambda d(x,y) + \frac{\mu d\left(x,F^mx\right)d\left(y,G^ny\right) + \gamma d\left(y,F^mx\right)d\left(x,G^ny\right)}{1+d(x,y)}$$

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then F and G have a unique common fixed point.

By setting $m = n$ and $F = G = T$ in Corollary (3.1.2), we deduce the following corollary.

Corollary 3.1.3. *[17] If $T : X \to X$ is a mapping defined on a complete complex valued metric space (X, d) satisfying the condition (for some fixed n):*

$$d\left(T^n x, T^n y\right) \precsim \lambda d(x, y) + \frac{\mu d\left(x, T^n x\right) d\left(y, T^n y\right) + \gamma d\left(y, T^n x\right) d\left(x, T^n y\right)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$, then T has a unique fixed point.

Corollary 3.1.4. *[17] If $T : X \to X$ is a mapping defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d\left(T^n x, T^n y\right) \precsim \lambda d(x, y)$$

for all $x, y \in X$, where λ, are nonnegative reals with $\lambda < 1$, then T has a unique fixed point.

Exemple 6. *Let $X = \mathbb{C}$ be the set of complex numbers. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d\left(z_1, z_2\right) = \left|x_1 - x_2\right| + i\left|y_1 - y_2\right|$ where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $(\mathbb{C}, d)$ is a complete complex valued metric space. Define $T : \mathbb{C} \to \mathbb{C}$ as*

$$T(x + iy) = \begin{cases} 0, & x, y \in \mathbb{Q} \\ 1 + i, & x, y \in \mathbb{Q}^c \\ 1, & x \in \mathbb{Q}^c, y \in \mathbb{Q} \\ i, & x \in \mathbb{Q}, y \in \mathbb{Q}^c \end{cases}$$

Now for $x = \dfrac{1}{\sqrt{2}}$ and $y = 0$ we get

$$d\left(T\left(\frac{1}{\sqrt{2}}\right), T(0)\right) = d(1, 0) = 1 \precsim \lambda d\left(\frac{1}{\sqrt{2}}, 0\right) = \lambda \frac{1}{\sqrt{2}}$$

Thus $\lambda \geq \sqrt{2}$, which is a contradiction as $0 \leq \lambda < 1$. However, notice that $T^2 z = 0$, so that

$$0 = d\left(T^2 z_1, T^2 z_2\right) \precsim \lambda d\left(z_1, z_2\right)$$

3.2 More Common Fixed Point Theorems

Theorem 3.2.1. *[17] Let (X, d) be a complete complex valued metric space wherein the mappings $S, T : X \to X$ satisfy the inequality*

$$d(Sx,Ty) \precsim \begin{cases} \lambda d(x,y) + \mu \dfrac{d(x,Sx)d(y,Ty) + d(y,Sx)d(x,Ty)}{d(Sx,x) + d(Ty,y)} \\ +\gamma \dfrac{d(x,Sx)d(x,Ty) + d(y,Sx)d(y,Ty)}{d(Sx,y) + d(Ty,x)}, \quad \text{if } D \neq 0, D_1 \neq 0 \\ 0, \quad \text{if } D = 0 \text{ or } D_1 = 0 \end{cases} \tag{3.2}$$

for all $x, y \in X$, where $D = d(Sx, x) + d(Ty, y)$ and $D_1 = d(Sx, y) + d(Ty, x)$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$. Then S, T have unique common fixed point

Proof. Let x_0 be an arbitrary point in X.

Define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \ldots$ Now, we distinguish two cases.

First, if $[d(Sx_{2k}, x_{2k}) + d(Tx_{2k+1}, x_{2k+1})][d(Sx_{2k}, x_{2k+1}) + d(Tx_{2k+1}, x_{2k})] \neq 0$, and

$[d(Sx_{2k+2}, x_{2k+2}) + d(Tx_{2k+1}, x_{2k+1})][d(Sx_{2k+2}, x_{2k+1}) + d(Tx_{2k+1}, x_{2k+2})] \neq 0$.

(for Any $k = 0, 1, 2, \ldots$), then

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) =& d(Sx_{2k}, Tx_{2k+1}) \precsim \lambda d(x_{2k}, x_{2k+1}) \\ &+ \mu \frac{d(x_{2k}, Sx_{2k}) d(x_{2k+1}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) d(x_{2k}, Tx_{2k+1})}{d(Sx_{2k}, x_{2k}) + d(Tx_{2k+1}, x_{2k+1})} \\ &+ \gamma \frac{d(x_{2k}, Sx_{2k}) d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k}) d(x_{2k+1}, Tx_{2k+1})}{d(Sx_{2k}, x_{2k+1}) + d(Tx_{2k+1}, x_{2k})}. \end{aligned}$$

Since $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$, therefore

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) \precsim& \lambda d(x_{2k}, x_{2k+1}) + \mu \frac{d(x_{2k}, x_{2k+1}) d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1}) d(x_{2k}, x_{2k+2})}{d(x_{2k+1}, x_{2k}) + d(x_{2k+2}, x_{2k+1})} \\ &+ \gamma \frac{d(x_{2k}, x_{2k+1}) d(x_{2k}, x_{2k+2}) \mid d(x_{2k+1}, x_{2k+1}) d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, x_{2k+1}) + d(x_{2k+2}, x_{2k})} \end{aligned}$$

or

$$d(x_{2k+1}, x_{2k+2}) \precsim \lambda d(x_{2k}, x_{2k+1}) + \mu \frac{d(x_{2k}, x_{2k+1}) d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, x_{2k}) + d(x_{2k+2}, x_{2k+1})} + \gamma \frac{d(x_{2k}, x_{2k+1}) d(x_{2k}, x_{2k+2})}{d(x_{2k+2}, x_{2k})}$$

so that

$$|d(x_{2k+1}, x_{2k+2})| \leq \lambda |d(x_{2k}, x_{2k+1})| + \mu \frac{|d(x_{2k}, x_{2k+1})| \cdot |d(x_{2k+1}, x_{2k+2})|}{|d(x_{2k+1}, x_{2k}) + d(x_{2k+2}, x_{2k+1})|} + \gamma |d(x_{2k}, x_{2k+1})|$$

Since

$$|d(x_{2k+1},x_{2k})+d(x_{2k+2},x_{2k+1})|\geq|d(x_{2k+1},x_{2k})|$$

therefore

$$|d(x_{2k+1},x_{2k+2})|\leq\lambda|d(x_{2k},x_{2k+1})|+\mu|d(x_{2k+1},x_{2k+2})|+\gamma|d(x_{2k},x_{2k+1})|$$

so that

$$|d(x_{2k+1},x_{2k+2})|\leq\frac{\lambda+\gamma}{1-\mu}|d(x_{2k},x_{2k+1})|.$$

Also

$$\begin{aligned}d(x_{2k+2},x_{2k+3})=&d(Sx_{2k+2},Tx_{2k+1})\precsim\lambda d(x_{2k+2},x_{2k+1})\\&+\mu\frac{d(x_{2k+2},Sx_{2k+2})\,d(x_{2k+1},Tx_{2k+1})+d(x_{2k+1},Sx_{2k+2})\,d(x_{2k+2},Tx_{2k+1})}{d(Sx_{2k+2},x_{2k+2})+d(Tx_{2k+1},x_{2k+1})}\\&+\gamma\frac{d(x_{2k+2},Sx_{2k+2})\,d(x_{2k+2},Tx_{2k+1})+d(x_{2k+1},Sx_{2k+2})\,d(x_{2k+1},Tx_{2k+1})}{d(Sx_{2k+2},x_{2k+1})+d(Tx_{2k+1},x_{2k+2})}\end{aligned}$$

Since $x_{2k+3}=Sx_{2k+2}$ and $x_{2k+2}=Tx_{2k+1}$, we get

$$\begin{aligned}d(x_{2k+2},x_{2k+3})\precsim\ &\lambda d(x_{2k+2},x_{2k+1})+\mu\frac{d(x_{2k+2},x_{2k+3})\,d(x_{2k+1},x_{2k+2})+d(x_{2k+1},x_{2k+3})\,d(x_{2k+2},x_{2k+2})}{d(x_{2k+3},x_{2k+2})+d(x_{2k+2},x_{2k+1})}\\&+\gamma\frac{d(x_{2k+2},x_{2k+3})\,d(x_{2k+2},x_{2k+2})+d(x_{2k+1},x_{2k+3})\,d(x_{2k+1},x_{2k+2})}{d(x_{2k+3},x_{2k+1})+d(x_{2k+2},x_{2k+2})}\end{aligned}$$

or

$$d(x_{2k+2},x_{2k+3})\precsim\lambda d(x_{2k+2},x_{2k+1})+\mu\frac{d(x_{2k+2},x_{2k+3})\,d(x_{2k+1},x_{2k+2})}{d(x_{2k+3},x_{2k+2})+d(x_{2k+2},x_{2k+1})}+\gamma\frac{d(x_{2k+1},x_{2k+3})\,d(x_{2k+1},x_{2k+2})}{d(x_{2k+1},x_{2k+3})}$$

so that

$$|d(x_{2k+2},x_{2k+3})|\leq\lambda|d(x_{2k+2},x_{2k+1})|+\mu\frac{|d(x_{2k+2},x_{2k+3})|\cdot|d(x_{2k+1},x_{2k+2})|}{|d(x_{2k+3},x_{2k+2})+d(x_{2k+2},x_{2k+1})|}+\gamma|d(x_{2k+1},x_{2k+2})|$$

Since

$$|d(x_{2k+3},x_{2k+2})+d(x_{2k+2},x_{2k+1})|\geq|d(x_{2k+1},x_{2k+2})|$$

therefore

$$|d(x_{2k+2},x_{2k+3})|\leq\lambda|d(x_{2k+2},x_{2k+1})|+\mu|d(x_{2k+2},x_{2k+3})|+\gamma|d(x_{2k+1},x_{2k+2})|$$

so

$$|d(x_{2k+2},x_{2k+3})| \leq \frac{\lambda+\gamma}{1-\mu}|d(x_{2k+1},x_{2k+2})|$$

Now, with $h=\dfrac{\lambda+\gamma}{1-\mu}$, we have (for all n)

$$\begin{aligned}|d(x_n,x_{n+1})| &\leq h\,|d(x_{n-1},x_n)|\\ &\leq \cdots \leq h^n\,|d(x_0,x_1)|\end{aligned}$$

So, for any $m>n$, we have

$$\begin{aligned}|d(x_n,x_m)| &\leq |d(x_n,x_{n+1})|+|d(x_{n+1},x_{n+2})|+\cdots+|d(x_{m-1},x_m)|\\ &\leq \left[h^n+h^{n+1}+\cdots+h^{m-1}\right]|d(x_0,x_1)|\\ &\leq \left[\frac{h^n}{1-h}\right]|d(x_0,x_1)|\end{aligned}$$

and henceforth

$$|d(x_n,x_m)| \leq \left[\frac{h^n}{1-h}\right]|d(x_0,x_1)| \to 0 \quad \text{as } m,n\to\infty.$$

We conclude that (x_n) is a Cauchy sequence. Since X is a complete, then there exists $l\in X$ such that $x_n\to l$ as $n\to\infty$. Now, we assert that $l=Sl$, otherwise $d(l,Sl)=z>0$ and we have

$$\begin{aligned}z=&d(l,Sl)\precsim d(l,Tx_{2k+1})+d(Tx_{2k+1},Sl)\\ &\precsim d(l,x_{2k+2})+\lambda d(l,x_{2k+1})+\mu\frac{d(l,Sl)d(x_{2k+1},Tx_{2k+1})+d(x_{2k+1},Sl)\,d(l,Tx_{2k+1})}{d(Sl,l)+d(Tx_{2k+1},x_{2k+1})}\\ &\quad+\gamma\frac{d(l,Sl)d(l,Tx_{2k+1})+d(x_{2k+1},Sl)\,d(x_{2k+1},Tx_{2k+1})}{d(Sl,x_{2k+1})+d(Tx_{2k+1},l)}\end{aligned}$$

which amounts to say that

$$\begin{aligned}|z|=&|d(l,Sl)|\leq |d(l,x_{2k+2})|+\lambda\,|d(l,x_{2k+1})|+\mu\frac{|z|\cdot|d(x_{2k+1},x_{2k+2})|+|d(x_{2k+1},Sl)|\cdot|d(l,x_{2k+2})|}{|d(Sl,l)+d(x_{2k+2},x_{2k+1})|}\\ &+\gamma\frac{|z|\cdot|d(l,x_{2k+2})|+|d(x_{2k+1},Sl)|\cdot|d(x_{2k+1},x_{2k+2})|}{|d(Sl,x_{2k+1})+d(x_{2k+2},l)|},\end{aligned}$$

a contradiction so that $|z|=|d(l,Sl)|=0$ i.e. $l=Sl$. It follows, similarly, that $l=Tl$. We now prove that S and T have a unique common fixed point. For this, assume that l^* in X is an another common fixed point of S and T. Then we have $Sl^*=Tl^*=l^*$

Since $D = d(Sl, l) + d(Tl^*, l^*) = 0$, therefore by definition of contraction condition

$d(l, l^*) = d(Sl, Tl^*) = 0$ so that $l = l^*$ which proves the uniqueness of common fixed point.

Second, we consider the case:(for any k)

$$(d(Sx_{2k}, x_{2k}) + d(Tx_{2k+1}, x_{2k+1})) \times (d(Sx_{2k}, x_{2k+1}) + d(Tx_{2k+1}, x_{2k})) = 0 \tag{3.3}$$

or

$$(d(Sx_{2k+2}, x_{2k+2}) + d(Tx_{2k+1}, x_{2k+1})) \times (d(Sx_{2k+2}, x_{2k+1}) + d(Tx_{2k+1}, x_{2k+2})) = 0 \tag{3.4}$$

if (3.3) is satisfy, This implies $d(Sx_{2k}, Tx_{2k+1}) = 0$.

so, $x_{2k} = Sx_{2k} = x_{2k+1} = Tx_{2k+1} = x_{2k+2}$. Thus, we have $x_{2k+1} = Sx_{2k} = x_{2k}$, so there exist n_1 and m_1 such that $n_1 = Sm_1 = m_1$. Using foregoing arguments, one can also show that there exist n_2 and m_2 such that $n_2 = Tm_2 = m_2$. As $d(Sm_1, m_1) + d(Tm_2, m_2) = 0$, (due to definition) implies $d(Sm_1, Tm_2) = 0$, so that $n_1 = Sm_1 = Tm_2 = n_2$ which in turn yields that $n_1 = Sm_1 = Sn_1$. Similarly, one can also have $n_2 = Tn_2$. As $n_1 = n_2$, implies $Sn_1 = Tn_1 = n_1$, therefore $n_1 = n_2$, is a common fixed point of S and T

We now prove that S and T have unique common fixed point. For this, assume that n_1^* in X is an another common fixed point of S and T. Then we have

$$Sn_1^* = Tn_1^* = n_1^*$$

Since $D = d(Sn_1, n_1) + d(Tn_1^*, n_1^*) = 0$, therefore

$$d(n_1, n_1^*) = d(\mathrm{Sn}_1, Tn_1^*) = 0$$

This implies that $n_1^* = n_1$

if (3.4) is satisfy, This implies $d(Sx_{2k+2}, Tx_{2k+1}) = 0$.

so, $x_{2k+1} = Tx_{2k+1} = x_{2k+2} = Sx_{2k+2} = x_{2k+3}$., then also proof can be completed by the same method. This completes the proof of the theorem. $\square$

By setting $S = T$, we get the following.

Corollary 3.2.1. *[17] Let (X, d) be a complete complex valued metric space and let the mappings $T : X \to X$ satisfy:*

$$d(Tx, Ty) \precsim \begin{cases} \lambda d(x, y) + \mu \dfrac{d(x, Tx)d(y, Ty) + d(y, Tx)d(x, Ty)}{d(Tx, x) + d(Ty, y)} \\ +\gamma \dfrac{d(x, Tx)d(x, Ty) + d(y, Tx)d(y, Ty)}{d(Tx, y) + d(Ty, x)}, \quad \text{if } D \neq 0, D_1 \neq 0 \\ 0, \quad \text{if } D = 0 \text{ or } D_1 = 0 \end{cases} \tag{3.5}$$

for all $x, y \in X$, where $D = d(Tx, x) + d(Ty, y)$ and $D_1 = d(Tx, y) + d(Ty, x)$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$. Then T has a unique fixed point.

As an application of Theorem (3.2.1), we prove the following theorem for two finite families of mappings.

Theorem 3.2.2. *[17] If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X, d) such that the mappings S and T (with $T = T_1T_2 \cdots T_m$ and $S = S_1S_2 \cdots S_n$) satisfy condition (3.2), then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof. The proof of this theorem is identical to that of Theorem (3.1.2). □

By setting $T_1 = T_2 = \cdots = T_m = F$ and $S_1 = S_2 = \cdots = S_n = G$, in Theorem (3.2.2), we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.2.2. *[17] If F and G are two commuting self-mappings defined on a complete complex valued metric space (X, d) satisfying the condition*

$$d(F^m x, G^n y) \precsim \begin{cases} \lambda d(x, y) + \mu \dfrac{d(x, F^m x)\, d(y, G^n y) + d(y, F^m x)\, d(x, G^n y)}{d(F^m x, x) + d(G^n y, y)} \\ +\gamma \dfrac{d(x, F^m x)\, d(x, G^n y) + d(y, F^m x)\, d(y, G^n y)}{d(F^m x, y) + d(G^n y, x)}, \quad \text{if } D \neq 0, D_1 \neq 0 \\ 0, \quad \text{if } D = 0 \text{ or } D_1 = 0 \end{cases}$$

for all $x, y \in X$, where $D = d(F^m x, x) + d(G^n y, y)$ and $D_1 = d(F^m x, y) + d({}^n y, x)$ and λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$. Then F,G have a unique common fixed point.

By setting $m=n$ and $F=G=T$ in Corollary (3.2.2), we deduce the following corollary.

Corollary 3.2.3. *[17] Let (X,d) be a complete complex valued metric space and let the mappings $T:X\to X$ satisfy (for some fixed n):*

$$d\left(T^n x, T^n y\right) \precsim \begin{cases} \lambda d(x,y)+\mu \dfrac{d\left(x,T^n x\right) d\left(y,T^n y\right)+d\left(y,T^n x\right) d\left(x,T^n y\right)}{d\left(T^n x,x\right)+d\left(T^n y,y\right)} & \\ +\gamma \dfrac{d\left(x,T^n x\right) d\left(x,T^n y\right)+d\left(y,T^n x\right) d\left(y,T^n y\right)}{d\left(T^n x,y\right)+d\left(T^n y,x\right)}, & \text{if } D\neq 0, D_1\neq 0 \\ 0, & \text{if } D=0 \text{ or } D_1=0 \end{cases} \tag{3.6}$$

for all $x,y\in X$ where λ,μ,γ are nonnegative reals with $\lambda+\mu+\gamma<1$ besides $D=d\left(T^n x,x\right)+d\left(T^n y,y\right)$ and $D_1=d\left(T^n x,y\right)+d\left(T^n y,x\right)$. Then T has a unique fixed point.

Exemple 7.

$$X_1=\{z\in\mathbb{C}:\operatorname{Re}(z)\geq 0, \operatorname{lm}(z)=0\}$$

$$X_2=\{z\in\mathbb{C}:\operatorname{Im}(z)\geq 0, \operatorname{Re}(z)=0\}$$

and write $X=X_1\cup X_2$. Define a mapping $d:X\times X\to\mathbb{C}$ as

$$d\left(z_1,z_2\right)=\begin{cases} i\left|x_1-x_2\right|, & z_1,z_2\in X_1 \\ \dfrac{2i}{3}\left|y_1-y_2\right|, & z_1,z_2\in X_2 \\ i\left(x_1+\dfrac{2}{3}y_2\right), & z_1\in X_1, z_2\in X_2 \\ i\left(x_2+\dfrac{2}{3}y_1\right), & z_1\in X_2, z_2\in X_1 \end{cases}$$

where $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, then (X,d) is a complete complex valued metric space. Define a self-mapping T on X (with $z=x+iy$) as

$$T(x+iy)=\begin{cases} 0, & x,y\in\mathbb{Q} \\ 1, & x\in\mathbb{Q}^c, y\in\mathbb{Q} \\ i, & x\in\mathbb{Q}, y\in\mathbb{Q}^c \end{cases}$$

By a routine calculation, one can verify that the map T^2 satisfies condition (3.6) *with* $\lambda = \mu = \gamma = \dfrac{1}{4}$ *(say). It is interesting to notice that this example cannot be covered by Corollary* (3.2.1) *as* $z_1 = 0, z_2 = \dfrac{1}{\sqrt{11}} \in X$ *implies*

$$\begin{aligned} i = & d(Tz_1, Tz_2) \precsim \lambda d(z_1, z_2) + \mu \frac{d(z_1, Tz_1)\, d(z_2, Tz_2) + d(z_2, Tz_1)\, d(z_1, Tz_2)}{d(Tz_1, z_1) + d(Tz_2, z_2)} \\ & + \gamma \frac{d(z_1, Tz_1)\, d(z_1, Tz_2) + d(z_2, Tz_1)\, d(z_2, Tz_2)}{d(Tz_1, z_2) + d(Tz_2, z_1)} \\ = & \lambda 0.3015i + \mu 0.4216i + \gamma 0.1617i \precsim 0.8848i \end{aligned}$$

a contradiction for every choice of λ, μ, γ *which amounts to say that condition* (3.5) *is not satisfied. Notice that the point* $0 \in X$ *remains fixed under* T *and* T^2 *and is indeed unique.*

Chapter 4

Generalized Common Fixed Point Theorems in Complex Valued Metric Spaces and Application

In this chapter, we will talk about some properties and special points in maps. We will also see a theorems in algebra that will help us in proving important theorems, which are a generalization of the theorems of the second chapter. and we will see an application to study the existence and uniquness of common solution of the system Urysohn integral equations.

This chapter is the the detail of the article [19].

4.1 Map's Properties and Special Points

Definition 4.1.1. *[19] Let S and T be self mappings of a nonempty set X.*

1. *A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and we shall called $w = Sx = Tx$ that a point of coincidence of S and T.*
2. *A point $x \in X$ is said to be a common fixed point of S and T if $x = Sx = Tx$.*

Definition 4.1.2. *[19] Let X be a non-empty set. The mappings S and T are commuting if*

$TSx = STx$ for all $x \in X$.

Definition 4.1.3. *[19] Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be weakly commuting if*

$$d(STx, TSx) \leq d(Sx, Tx)$$

for all $x \in X$.

Definition 4.1.4. *[19] Let S and T be mappings from a metric space (X, d) into itself. The mapping S and T are said to be compatible if $\lim_{n\to+\infty} d(STx_n, TSx_n) = 0$ whenever (x_n) is a sequence in X such that $\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Tx_n = z$ for some $z \in X$*

Remark 4.1.1. *In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true.*

Exemple 8. *[16] Let $X = [0, 1]$ with the usual metric. Define $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{x}{2+x}$. Then, for all x in X, one obtains*

$$d(fgx, gfx) = \frac{x}{4+x} - \frac{x}{4+2x} = \frac{x^2}{(4+x)(4+2x)} \leq \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(fx, gx)$$

f *and* g *are weakly commuting. But, for any non-zero $x \in X$, we have*

$$gfx = \frac{x}{4+x} > \frac{x}{4+2x} = fgx$$

f *and g do not commute.*

Exemple 9. *[13] Let $X = [0, 1]$ with the usual metric. Define $f(x) = x^3$ and $g(x) = 2x^3$. is clear that f and g are not weakly commuting but are compatible.*

Definition 4.1.5. *[19] Let S and T be self mappings of a nonempty set X. The mapping S and T are weakly compatible if $STx = TSx$ whenever $Sx = Tx$.*

Remark 4.1.2. *Every compatible mapping are weakly compatible,but the convers is not necessarily true.*

Exemple 10. *[7] Let $X = [2, 20]$ with the usual metric. Define*

$$A(x) = \begin{cases} 2 & x = 2 \\ 13 + x & , 2 < x \leq 5 \\ x - 3 & x > 5 \end{cases}$$

$$S(x) = \begin{cases} 2 & , x \in \{2\} \cup]5, 20] \\ 8 & , 2 < x \leq 5 \end{cases}$$

Let (x_n) be the sequence defined by $x_n = 5 + \frac{1}{n}, n \geq 1$. Then Clearly A and S are weakly compatible maps, but are not compatible, because

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = 2$$

but

$$\lim_{n \to +\infty} ASx_n = 2 \neq 8 = \lim_{n \to +\infty} SAx_n$$

Lemma 4.1.1. *[9] Let X be a nonempty set and $T : X \longrightarrow X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \longrightarrow X$ is one-to-one.*

Proof. Proof. Define a multifunction, $F : T(X) \to 2^X$,by $F(y) = \{x \in X : T(x) = y\}$.
By using the **axiom of choice**, F has a selector that is, there is a function $g : T(X) \to X$ such that $g(y) \in F(y)$ for all $y \in T(X)$. Note that, $T(g(y)) = y$ for all $y \in T(X)$. Now, put $E = \{g(y) : y \in T(X)\}$. It is clear that T is one-to-one on E and $T(E) = T(X)$. □

4.2 Common Fixed Points Theorems

Theorem 4.2.1. *[19] Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If there exists a mapping $\Lambda, \Xi : X \to [0, 1[$ such that for all $x, y \in X$:*

1. *$\Lambda(Sx) \leq \Lambda(x)$ and $\Xi(Sx) \leq \Xi(x)$.*

2. $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$

3. $(\Lambda + \Xi)(x) < 1.$

4. $d(Sx, Ty) \precsim \Lambda(x)d(x, y) + \dfrac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)}.$

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $S(X) \subseteq X$ and $T(X) \subseteq X$, we can construct the sequence $\{x_k\}$ in X such that

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1} \tag{4.1}$$

for all $k \geq 0$. From hypothesis and (4.1) we get

$$\begin{aligned}
|d(x_{2k+1}, x_{2k+2})| &= |d(Sx_{2k}, Tx_{2k+1})| \\
&\leq \Lambda(x_{2k})|d(x_{2k}, x_{2k+1})| + \frac{\Xi(x_{2k})|d(x_{2k}, Sx_{2k})||d(x_{2k+1}, Tx_{2k+1})|}{|1 + d(x_{2k}, x_{2k+1})|} \\
&= \Lambda(x_{2k})|d(x_{2k}, x_{2k+1})| + \frac{\Xi(x_{2k})|d(x_{2k}, x_{2k+1})||d(x_{2k+1}, x_{2k+2})|}{|1 + d(x_{2k}, x_{2k+1})|} \\
&= \Lambda(x_{2k})|d(x_{2k}, x_{2k+1})| + \Xi(x_{2k})|d(x_{2k+1}, x_{2k+2})| \left(\frac{|d(x_{2k}, x_{2k+1})|}{|1 + d(x_{2k}, x_{2k+1})|}\right) \\
&\leq \Lambda(x_{2k})|d(x_{2k}, x_{2k+1})| + \Xi(x_{2k})|d(x_{2k+1}, x_{2k+2})| \\
&= \Lambda(Tx_{2k-1})|d(x_{2k}, x_{2k+1})| + \Xi(Tx_{2k-1})|d(x_{2k+1}, x_{2k+2})| \\
&\leq \Lambda(x_{2k-1})|d(x_{2k}, x_{2k+1})| + \Xi(x_{2k-1})|d(x_{2k+1}, x_{2k+2})| \\
&= \Lambda(Sx_{2k-2})|d(x_{2k}, x_{2k+1})| + \Xi(Sx_{2k-2})||d(x_{2k+1}, x_{2k+2})| \\
&\leq \Lambda(x_{2k-2})|d(x_{2k}, x_{2k+1})| + \Xi(x_{2k-2})|d(x_{2k+1}, x_{2k+2})| \\
&\vdots \\
&\leq \Lambda(x_0)|d(x_{2k}, x_{2k+1})| + \Xi(x_0)|d(x_{2k+1}, x_{2k+2})|,
\end{aligned}$$

which is implies that

$$|d(x_{2k+1}, x_{2k+2})| \leq \left(\frac{\Lambda(x_0)}{1 - \Xi(x_0)}\right)|d(x_{2k}, x_{2k+1})|.$$

Similarly, we get

$$
\begin{aligned}
|d(x_{2k+2}, x_{2k+3})| &= |d(x_{2k+3}, x_{2k+2})| \\
&= |d(Sx_{2k+2}, Tx_{2k+1})| \\
&\leq \Lambda(x_{2k+2})|d(x_{2k+2}, x_{2k+1})| + \frac{\Xi(x_{2k+2})|d(x_{2k+2}, Sx_{2k+2})||d(x_{2k+1}, Tx_{2k+1})|}{|1 + d(x_{2k+2}, x_{2k+1})|} \\
&= \Lambda(x_{2k+2})|d(x_{2k+2}, x_{2k+1})| + \frac{\Xi(x_{2k+2})|d(x_{2k+2}, x_{2k+3})||d(x_{2k+1}, x_{2k+2})|}{|1 + d(x_{2k+1}, x_{2k+2})|} \\
&= \Lambda(x_{2k+2})|d(x_{2k+2}, x_{2k+1})| + \Xi(x_{2k+2})|d(x_{2k+2}, x_{2k+3})|\left(\frac{|d(x_{2k+2}, x_{2k+1})|}{|1 + d(x_{2k+1}, x_{2k+2})|}\right) \\
&\leq \Lambda(x_{2n+2})|d(x_{2k+2}, x_{2k+1})| + \Xi(x_{2k+2})|d(x_{2k+2}, x_{2k+3})| \\
&= \Lambda(Tx_{2k+1})|d(x_{2k+2}, x_{2k+1})| + \Xi(Tx_{2k+1})|d(x_{2k+2}, x_{2k+3})| \\
&\leq \Lambda(x_{2n+1})|d(x_{2k+2}, x_{2k+1})| + \Xi(x_{2k+1})|d(x_{2k+2}, x_{2k+3})| \\
&= \Lambda(Sx_{2k})|d(x_{2k+2}, x_{2k+1})| + \Xi(Sx_{2k})|d(x_{2k+2}, x_{2k+3})| \\
&\leq \Lambda(x_{2k})|d(x_{2k+2}, x_{2k+1})| + \Xi(x_{2k})|d(x_{2k+2}, x_{2k+3})| \\
&\vdots \\
&\leq \Lambda(x_0)|d(x_{2k+2}, x_{2k+1})| + \Xi(x_0)|d(x_{2k+2}, x_{2k+3})| \\
&= \Lambda(x_0)|d(x_{2k+1}, x_{2k+2})| + \Xi(x_0)|d(x_{2k+2}, x_{2k+3})|,
\end{aligned}
$$

which is implies that

$$|d(x_{2k+2}, x_{2k+3})| \leq \left(\frac{\Lambda(x_0)}{1 - \Xi(x_0)}\right)|d(x_{2k+1}, x_{2k+2})|.$$

Now, we set $\alpha = \dfrac{\Lambda(x_0)}{1 - \Xi(x_0)}$, it follows that

$$
\begin{aligned}
|d(x_n, x_{n+1})| &\leq \alpha|d(x_{n-1}, x_n)| \\
&\leq \alpha^2|d(x_{n-2}, x_{n-1})| \\
&\vdots \\
&\leq \alpha^n|d(x_0, x_1)|
\end{aligned}
$$

for all $n \in \mathbb{N}$. Now, for any positive integer m and n with $m > n$, we have

$$\begin{aligned}
|d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\
&\leq \alpha^n |d(x_0, x_1)| + \alpha^{n+1} |d(x_0, x_1)| + \cdots + \alpha^{m-1} |d(x_0, x_1)| \\
&= \left(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}\right) |d(x_0, x_1)| \\
&\leq \left(\frac{\alpha^n}{1-\alpha}\right) |d(x_0, x_1)|.
\end{aligned}$$

Therefore,

$$|d(x_n, x_m)| \leq \left(\frac{\alpha^n}{1-\alpha}\right) |d(x_0, x_1)|.$$

Since $\alpha \in [0, 1[$, if we taking limit as $m, n \to 0$, then $|d(x_n, x_m)| \to 0$, which implies that (x_n) is a Cauchy sequence. By completeness of X, there exists a point $z \in X$ such that $x_k \to z$ as $k \to \infty$. Next, we claim that $Sz = z$. By the notion of a complex valued metric d, we have

$$\begin{aligned}
d(z, Sz) &\precsim d(z, x_{2k+2}) + d(x_{2k+2}, Sz) \\
&= d(z, x_{2k+2}) + d(Tx_{2k+1}, Sz) \\
&= d(z, x_{2k+2}) + d(Sz, Tx_{2k+1}) \\
&\precsim d(x_{2k+2}, z) + \Lambda(z) d(z, x_{2k+1}) + \frac{\Xi(z) d(z, Sz) d(x_{2k+1}, Tx_{2k+1})}{1 + d(z, x_{2k+1})} \\
&= d(x_{2k+2}, z) + \Lambda(z) d(z, x_{2k+1}) + \frac{\Xi(z) d(z, Sz) d(x_{2k+1}, x_{2k+2})}{1 + d(z, x_{2k+1})}
\end{aligned}$$

which implies that

$$|d(z, Sz)| \leq |d(x_{2k+2}, z)| + \Lambda(z) |d(z, x_{2k+1})| + \frac{\Xi(z) |d(x_{2k+1}, x_{2k+2})| |d(z, Sz)|}{1 + |d(z, x_{2k+1})|}.$$

Taking $k \to \infty$, we have $|d(z, Sz)| = 0$, which implies that $d(z, Sz) = 0$. Thus, we get $z = Sz$. It follows similarly that $z = Tz$. Therefore, z is a common fixed point of S and T. Finally, we show that z is a unique common fixed point of S and T. Assume that there exists another common fixed point z_1 that is $z_1 = Sz_1 = Tz_1$. It follows from

$$\begin{aligned}
d(z, z_1) &= d(Sz, Tz_1) \\
&\precsim \Lambda(z) d(z, z_1) + \frac{\Xi(z) d(z, Sz) d(z_1, Tz_1)}{1 + d(z, z_1)} \\
&= \Lambda(z) d(z, z_1)
\end{aligned}$$

That $|d(z, z_1)| \leq \Lambda(z)\, |d(z, z_1)|$.

Since $\Lambda(z) \in [0, 1[$, we have $|d(z, z_1)| = 0$. Therefore, we have $z = z_1$ and thus z is a unique common fixed point of S and T. $\square$

Corollary 4.2.1. *[19] Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If S and T satisfy*

$$d(Sx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof. We can prove this result by applying Theorem (4.2.1)

by setting $\Lambda(x) = \lambda$ and $\Xi(x) = \mu$. $\square$

Corollary 4.2.2. *[19] Let (X, d) be a complete complex valued metric space and $T : X \to X$. If there exists a mapping $\Lambda, \Xi : X \to [0, 1[$ such that for all $x, y \in X$:*

1. $\Lambda(Tx) \leq \Lambda(x)$ *and* $\Xi(Tx) \leq \Xi(x)$;
2. $(\Lambda + \Xi)(x) < 1$;
3. $d(Tx, Ty) \precsim \Lambda(x) d(x, y) + \dfrac{\Xi(x) d(x, Tx) d(y, Ty)}{1 + d(x, y)}$.

Then T has a unique fixed point.

Proof. We can prove this result by applying Theorem (4.2.1) with $S = T$. $\square$

Corollary 4.2.3. *[19] Let (X, d) be a complete complex valued metric space and $T : X \to X$. If T satisfies*

$$d(Tx, Ty) \precsim \lambda d(x, y) + \frac{\mu d(x, Tx) d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Corollary (4.2.2) with $\Lambda(x) = \lambda$ and $\Xi(x) = \mu$. $\square$

Theorem 4.2.2. *[19] Let (X, d) be a complete complex valued metric space and $T : X \to X$. If there exists a mapping $\Lambda, \Xi : X \to [0, 1[$ such that for all $x, y \in X$ and for some $n \in \mathbb{N}$*

1. $\Lambda(T^n x) \leq \Lambda(x)$ *and* $\Xi(T^n x) \leq \Xi(x)$;
2. $(\Lambda + \Xi)(x) < 1$;
3. $d(T^n x, T^n y) \precsim \Lambda(x) d(x, y) + \dfrac{\Xi(x) d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}$.

Then T has a unique fixed point.

Proof. From Corollary (4.2.2), we get T^n has a unique fixed point z. It follows from

$$T^n(Tz) = T(T^n z) = Tz$$

that Tz is a fixed point of T^n. Therefore $Tz = z$ by the uniqueness of a fixed point of T^n and then z is also a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. □

Corollary 4.2.4. *[19] Let (X, d) be a complete complex valued metric space and $S, T : X \to X$. If T satisfy*

$$d(T^n x, T^n y) \precsim \lambda d(x, y) + \frac{\mu d(x, T^n x) d(y, T^n y)}{1 + d(x, y)}$$

for all $x, y \in X$ for some $n \in \mathbb{N}$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then T has a unique fixed point.

Proof. We can prove this result by applying Theorem (4.2.2) with $\Lambda(x) = \lambda$ and $\Xi(x) = \mu$. Next, we prove a common fixed point theorem for weakly compatible mappings in complex valued metric spaces. □

Theorem 4.2.3. *[19] Let (X, d) be a complex valued metric space, $S, T : X \to X$ such that $T\ (X) \subseteq S(X)$ and $S(X)$ is complete. If there exists two mappings $\Lambda, \Xi : X \to [0, 1[$ such that for all $x, y \in X$*

1. *$\Lambda(Tx) \leq \Lambda(Sx)$ and $\Xi(Tx) \leq \Xi(Sx)$;*
2. *$(\Lambda + \Xi)(Sx) < 1$;*
3. *$d(Tx, Ty) \precsim \Lambda(Sx)d(Sx, Sy) + \dfrac{\Xi(Sx)d(Sx, Tx)d(Sy, Ty)}{1 + d(Sx, Sy)}$.*

Then S and T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Proof. By Lemma (4.1.1), there exists $E \subseteq X$ such that $S(E) = S(X)$ and $S : E \to X$ is one-to-one. Since

$$T(E) \subseteq T(X) \subseteq S(X) = S(E)$$

we can define a mapping $\Theta : S(E) \to S(E)$ by

$$\Theta(Sx) = Tx \tag{4.2}$$

Since S is one-to-one on E, then Θ is well-defined. From (*1*) and (4.2), we have

$$\Lambda(\Theta(Sx)) \leq \Lambda(Sx) \text{ and } \Xi(\Theta(Sx)) \leq \Xi(Sx) \tag{4.3}$$

From (*3*) and (4.2), we get

$$d(\Theta(Sx), \Theta(Sy)) \precsim \Lambda(Sx)d(Sx, Sy) + \frac{\Xi(Sx)d(Sx, \Theta(Sx))d(Sy, \Theta(Sy))}{1 + d(Sx, Sy)} \tag{4.4}$$

for all $Sx, Sy \in S(E)$. From $S(E) = S(X)$ is complete and (4.3) and (4.4) are holds, we use Corollary (4.2.2) with a mapping Θ, then there exists a unique fixed point $z \in S(X)$ such that $\Theta z = z$. Since $z \in S(X)$, we have $z = Sw$ for some $w \in X$. So $\Theta(Sw) = Sw$ that is $Tw = Sw$.

Therefore, T and S have a unique point of coincidence. Next, we claim that S and T have a common fixed point. Since S and T are weakly compatible and $z = Tw = Sw$, we get

$$Sz = STw = TSw = Tz$$

Hence $Sz = Tz$ is a point of coincidence of S and T. Since z is the only point of coincidence of S and T, we get $z = Sz = Tz$ which implies that z is a common fixed point of S and T. Finally, we show that z is a unique common fixed point of S and T. Assume that t be another common fixed point that is

$$t = St = Tt$$

Thus t is also a point of coincidence of S and T. However, we know that z is a unique point of coincidence of S and T. Therefore, we get $t = z$ that is z is a unique common fixed point of S and T. $\square$

4.3 Application

In this section, we apply Theorem (4.2.1) to the existence of common solution of the system of Urysohn integral equations.

Theorem 4.3.1. *[19]*

Let $X = \mathbb{C}([a,b], \mathbb{R}^n)$, where $[a,b] \subseteq \mathbb{R}^+$ and $d : X \times X \to \mathbb{C}$ is define by:

$$d(x,y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}.$$

Consider the Urysohn integral equations

$$x(t) = \int_a^b K_1(t,s,x(s))ds + g(t) \tag{4.5}$$

$$x(t) = \int_a^b K_2(t,s,x(s))ds + h(t) \tag{4.6}$$

where $t \in [a,b] \subset \mathbb{R}$ *and* $x, g, h \in X$. *Suppose that* $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *are such that* $F_x, G_x \in X$ *for all* $x \in X$, *where*

$$F_x(t) = \int_a^b K_1(t, s, x(s))ds$$

and

$$G_x(t) = \int_a^b K_2(t, s, x(s))ds$$

for all $t \in [a,b]$. *If there exists two mappings* $\Lambda, \Xi : X \to [0,1[$ *such that for all* $x, y \in X$ *the following holds:*

1. $\Lambda(F_x + g) \leq \Lambda(x)$ *and* $\Xi(F_x + g) \leq \Xi(x)$;

2. $\Lambda(G_x + h) \leq \Lambda(x)$ *and* $\Xi(G_x + h) \leq \Xi(x)$;

3. $(\Lambda + \Xi)(x) < 1$;

4. $\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a} \precsim \Lambda(x) A(x,y)(t) + \Xi(x) B(x,y)(t)$,

   where

$$A(x,y)(t) = \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$$

$$B(x,y)(t) = \frac{\|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x,y)} \sqrt{1+a^2} e^{i \tan^{-1} a}$$

then the system of integral Equations (4.5) *and* (4.6) *have a unique common solution.*

Proof. It is easily to check that (X, d) is a complex valued metric space. Define two mappings $S, T : X \times X \to X$ by $Sx = F_x + g$ and $Tx = G_x + h$. Then

$$d(Sx, Ty) = \max_{t \in [a,b]} \|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$$

$$d(x, Sx) = \max_{t \in [a,b]} \|F_x(t) + g(t) - x(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}$$

and

$$d(y, Ty) = \max_{t \in [a,b]} \|G_y(t) + h(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \tan^{-1} a}.$$

It is easily seen that for all $x, y \in X$, we have

1. $\Lambda(Sx) \leq \Lambda(x)$ and $\Xi(Sx) \leq \Xi(x)$;

2. $\Lambda(Tx) \leq \Lambda(x)$ and $\Xi(Tx) \leq \Xi(x)$;

3. $d(Sx, Ty) \precsim \Lambda(x)d(x, y) + \dfrac{\Xi(x)d(x, Sx)d(y, Ty)}{1 + d(x, y)}$.

By Theorem (4.2.1), we get S and T have a common fixed point. Thus there exists a unique point $x \in X$ such that $x = Sx = Tx$. Now, we have

$$x = Sx = F_x + g$$

and

$$x = Tx = G_x + h$$

that is

$$x(t) = \int_a^b K_1(t, s, x(s))ds + g(t)$$

and

$$x(t) = \int_a^b K_2(t, s, x(s))ds + h(t)$$

Therefore, we can conclude that the Urysohn integral (4.5) and (4.6) have a unique common fixed point. $\square$

Abstract

In this note, we have discussed the definition of complex valued metric spaces, and study of the existence and uniqueness of common fixed points.

In the first chapter we reminded of the metric space and Banach's theory of the fixed point.

In the second chapter we studied theorems of the existence and uniqueness of the common fixed points in complex valued metric spaces.

We made generalizations These theorems in the third and fourth chapters and we have applied one of these theorems to prove the existence and uniqueness of common solution of the system of Urysohn integral equations

Résumé

Dans cette mémoire, nous avons discuté la définition des espaces métriques à valeurs complexes, et l'étude de l'existence et l'unicité des points fixes communs.

Dans le premier chapitre nous avons rappelé l'espace métrique et la théorie de Banach du point fixe.

Dans la deuxième chapitre, nous avons étudié les théories de l'existence et de l'unicité des points fixes communs dans les espaces métriques à valeurs complexes.

Nous avons fait des généralisations des cettes théoremes dans le troisième et quatrième chapitres, et nous avons appliqué l'un de ces théorèmes pour étudier l'existence et l'unicité de solution commune du système d'équations intégrales d'Urysohn.

ملخص

لقد تطرقنا في هذه المذكرة الى التعريف بالفضاءات المترية ذات القيم العقدية، و دراسة وجود و وحدانية النقط الصامدة المشتركة.
ففي الفصل الأول ذكّرنا بالفضاء المتري و نظرية بناخ للنقطة الصامدة.
أما في الفصل الثاني درسنا نظريات وجود و وحدانية النقط الصامدة المشتركة.
و قد قمنا بتعميمات لهاته النظريات في الفصلين الثالث و الرابع مع تقديم تطبيق لإثبات وجود و وحدانية حل جملة معادلتين تكامليتين لاوريشون.

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