Galerkin method for the higher dimension Boussinesq equation non linear with integral condition

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Abstract-This paper deals with the solvability of a higher dimension mixed non local problem for a Boussinesq equation non linear. Galerkin's method was the main used tool for proving the solvability of the given non local problem.
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1 Introduction

By applying mathematical modeling to various phenomena of physics, biology and ecology there often arise problems with non-classical boundary conditions, which connect the values of the unknown function on the boundary and inside of the given domain. Some times the physical phenomena are modeled by non classical boundary value problems which involve a boundary condition as an integral condition over the spatial domain of a function of the desired solution. The nonlocal boundary condition arises mainly when the data on the boundary cannot be measured directly, but their average values are known. In the very recent years, nonlocal problems, particularly those with integral constraints have received great attention. The physical significance of nonlocal conditions such as a mean, total mass, moments, etc, has served as a fundamental cause for the considerably increasing interest to this kind of boundary value problems. Nonlocal problems are generally encountered in chemical engineering, heat transmission, plasma physics, heat transmission, thermoelsticity and underground water flow. See in this regard the papers by Ewing and Lin [3], Choi and Chan [2]. As a special application see Bouziani [1], where the author has considered a nonlocal problem which is proposed in the mathematical modeling of technologic process of external elimination of gas, practices in the refining of impurities of Silicon lamina.

In section 1, we state the problem, define some spaces and give a relevant definition of weak solution. Section 2 is devoted to the study of existence of the weak solution of the posed problem by applying Galerkin's method.

In this paper, we are concerned with the following nonlocal mixed boundary value problem for the $n-$dimensional Boussinesq equation non linear in a cylinder $Q_T = \Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.

$$\begin{cases} u_{tt} - \alpha^2 \Delta u - \beta^2 \Delta u_{tt} = |u|^{p-2} u, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \\ \frac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi,\tau) d\xi d\tau, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $p>2$, $\varphi(x)$ and $\psi(x)$ are given functions and $\frac{\partial u}{\partial \eta}$ designates the normal derivative.
Now let $V(Q_T)$ and $W(Q_T)$ be the set spaces defined respectively by:

$$V(Q_T)=\left\{u\in W_2^1(Q_T):\nabla u_t\in L^2(Q_T),u\in L^p(Q_T),\ u_t\in L^p(Q_T)\right\},$$

and

$$W(Q_T)=\{u\in V(Q_T):\ v(x,T)=0\}. \tag{2}$$

Consider the equation

$$(u_{tt},v)_{L^2(Q_T)}-\alpha^2(\Delta u,v)_{L^2(Q_T)}-\beta^2(\Delta u_{tt},v)_{L^2(Q_T)}=\left(|u|^{p-2}u,v\right)_{L^2(Q_T)}. \tag{3}$$

Evaluation of the inner products in (3) and use of boundary condition in (1) leads

$$\begin{aligned}&-(u_t,v_t)_{L^2(Q_T)}+\alpha^2(\nabla u,\nabla v)_{L^2(Q_T)}-\beta^2(\nabla u_t,\nabla v_t)_{L^2(Q_T)}\\
=&\left(|u|^{p-2}u,v\right)_{L^2(Q_T)}-(\psi(x),v(x,0))_{L^2(\Omega)}+\alpha^2\int_{\partial\Omega}\int_0^T v(x,t)\left(\int_0^t\int_\Omega u(\xi,\tau)d\xi\right)dtds_x\\
&+\beta^2\int_{\partial\Omega}\int_0^T v(x,t)\left(\int_\Omega u_t(\xi,t)d\xi\right)dtds_x-\beta^2\int_{\partial\Omega}\int_0^T v(x,t)\left(\int_\Omega u_t(\xi,0)d\xi\right)dtds_x\\
&+\beta^2(\nabla\psi(x),\nabla v(x,0))_{L^2(\Omega)},\end{aligned} \tag{4}$$

$\forall v\in W(Q_T)$.
Definition 1.1. A function $u\in V(Q_T)$ is called a generalized solution of problem (1), if it satisfies equation (4) for each $v\in W(Q_T)$ and $u(x,0)=\varphi(x)$.

2 Solvability of the problem

We now give the main result on the existence of solution of problem (1) and prove it by using the Galerkin method.

Theorem 1 *If $\varphi(x)\in W_2^1(\Omega)$, $\psi(x)\in L^p(\Omega)$ and $\psi(x,t)\in W_2^1(\Omega)$, then there is at least one generalized solution in $V(Q_T)$ to problem* (1).

References

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